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Hony. Joint Secretary

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PROGRESS REPORT.

The following gentlemen have been elected members of the Society—

1. *Mr. Dattatraya Ganesh Dandekar B.Sc.*—Science Master, Herbert High School, Kotah (Rajputana).

2. *Mr. V. Viraswamaiya B.A., L.T.*,—Assistant, Pachaiyappa's High School, 29, Krishnappa Naick Agraharam, Madras, (at concessional rate).

3. *Mr. Keshav Narasimha Khandekar, B.A* —Assistant Head Master, Mission High School, Beawar (Rajputana), (at concessional rate).

4. *Mr. F. Hallberg.*—Professor of Mathematics, St. Xavier's College, Fort Bombay.

5. *Mr. T. R. Venkatesa Aiyar, B.A.*—Editor, The Indian Engineer, 1/33, Kachaleswara Agraharam, George Town, Madras.

2. The Calcutta University Calendar for 1915, Part I, has been received for the Library.

POONA,
31st Jan., 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

Green's Function.

BY P. V. SESHU AIYAR.

NOTE: In this article Δ stands for the usual symbol ∇^2.

1. *Introduction* :

As a means of solving certain problems in Electrostatics Green introduced a certain function into Analysis which afterwards came to be known after his name. Properties of this function were developed by Green himself from a physical point of view (*vide* Green's *Mathematical Papers*, edited by Ferrers: p. 31 and seq.; or Clarke Maxwell's *Electricity and Magnetism*: p. 133 and seq.]. But continental mathematicians have developed these properties from the point of view of Pure Analysis and the present article gives the exposition from the latter point of view following French mathematicians, chiefly Poincaré.

2. *Definition* :

Let T be a volume bounded by a closed surface S, and M′ a point situated in the interior of T, and M a variable point in the interior of T or upon the surface S. Then the function $G=(1/r+H)$ is called the Function of Green relative to the volume T and to the point M′, where r is the distance of the variable point M from the point M′ and H is a function which is *harmonic* in the volume T (*i.e.*, satisfies the equation of Laplace with the necessary conditions as regards continuity) and which is equal to $-1/r$ upon S.

Thus defined, this function G vanishes upon S, and satisfies the equation of Laplace at every point of volume T except at M′ where it becomes infinite.

The function of Green given by this definition relates to the space enclosed by a closed surface and and a point within it corresponds to the interior problem of Dirichlet [*vide*: the article on the *Problem of Dirichlet* (p. 177) in the issue of October 1915 of this Journal]. As in the case of the problem of Dirichlet, there is also an exterior function of Green, *i.e.*, a function relative to a volume T and to a point M′ outside this volume. In this article we confine ourselves to the interior function.

3. *Physical Interpretation* :

Conceive the surface S to be a perfect conductor put in communication with the earth, and a unit of positive electricity placed at the point M′. Then the value at the variable point M of the total potential function, arising from the unit of electricity at M′ and from the electricity it will induce upon the surface, will be the function of Green as defined above.

For, in consequence of the communication with the earth, the total potential at the surface is zero; further the potential at M due to the unit of electricity placed at M′ is $1/r$ and that due to the electricity induced on the surface is a harmonic function H; and hence the total potential is $H+1/r=G$.

This physical interpretation also helps to convince us that such a function exists.

4. *Notation:*

Once the domain T is given, the function G is a function of the co-ordinates (x, y, z) of the variable point M, but it depends also on the co-odinates (x', y', z') of the point M′, the infinity of the function. To put in evidence this fact we may write it as $G(x, y, z, x', y', z')$, or G (M, M′) indicating by the latter notation, the value at the point M of the function of Green relative to the domain T and to the point M′.

5. *Properties of the Function*:—

1°. The function is positive at every point M in the domain T.

As r tends to zero (*i.e.*, as the point M tends to the point M′) H remains finite. Therefore $G-1/r$ remains finite.

Hence $(r.G-1)$ tends to zero, and rG tends to unity. Thus when r is very small, the product rG is very near to unity. We can therefore affirm that in the neighbourhood of the point M′, the function G is positive. Enclose this point by a very small sphere Σ. Between Σ and S, we have $\Delta G=0$; upon S, G is zero and upon Σ, $G>0$. Therefore between Σ and S, G is everywhere positive according to the following lemma.

Lemma:

Every harmonic function in a domain lies between the greatest and the least values upon the surface bounding the domain, whether it be simply-connected or multiply-connected. For, otherwise, there must be a point within the domain T, at which the function is a maximum or a minimum, which it cannot be [*vide*: the article on 'Dirichlet's Problem' quoted above.]

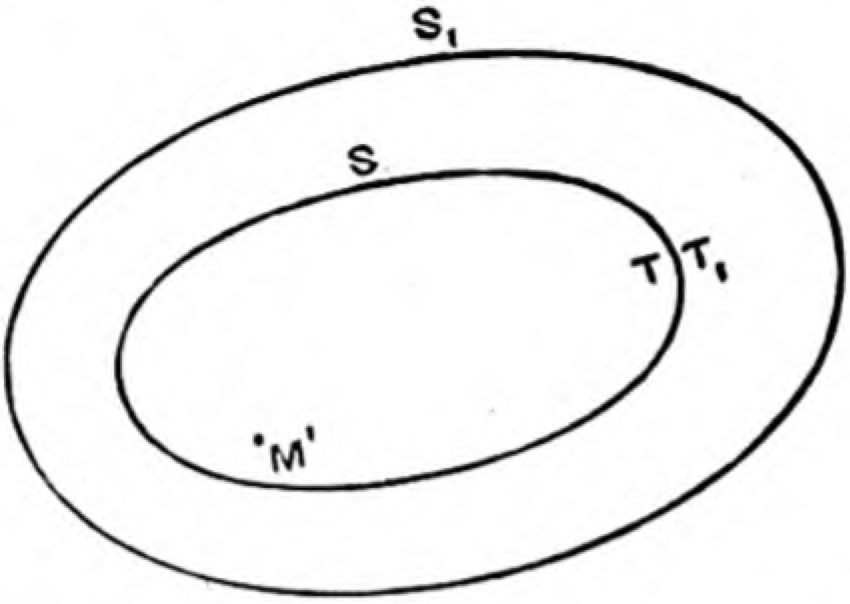
2°. In the volume T, G is everywhere less than $1/r$.

In the volume T, $\Delta H=0$ and $H=-1/r$ upon S. That is, H is negative upon S.

Since at any point within T, H must lie between its greatest and least values upon S, H is negative at every point of T.

$$\therefore \quad G=H+\frac{1}{r}<\frac{1}{r}$$

3°. Let T be a domain bounded by a closed surface S, and T_1 a larger domain bounded by S_1 and containing T within it; and let M′ be a point in T, then if G_1 be the function of Green relative to T_1 and to M′, and G that relative to T and to the same point M′, then, within T, $G_1 > G$.



For, if G_1 be $H_1 + 1/r$ and G be $H + 1/r$, we see that $G_1 - G = H_1 - H$ is harmonic within T and since $G_1 > 0$ in T_1, we have $H_1 + 1/r > 0$ upon S but upon S, $H = -1/r$; therefore $H_1 - H > 0$ upon S and is positive in the volume T enclosed by S according to the Lemma.

Hence $\qquad G_1 - G$ is positive;

i.e. $\qquad G_1 > G$ in T.

4.° Consider the surface $G = C$.

To each value of the constant C there corresponds a particular surface.

For very great values of the constant, we have evidently very small surfaces sorrounding the pole M′, since G becomes infinite at M′.

For values very small of C, we have on the contrary surfaces very near to the surface S.

Finally, we pass from one surface to another by a continuous deformation by making the constant C vary in a continuous manner.

Further the function G being uniform in the volume T, two surfaces corresponding to two different values of C cannot cut each other. The surfaces $G = C$, for different values of C, enclose therefore the pole and are mutually enveloping; that is, they are contained within one another, so that the surface corresponding to any one value of C (say C_o) encloses within it all surfaces corresponding to the values of C greater than C_o.

The physical interpretation of all the above mentioned properties is quite plain.

5°. $\int_{S'} \frac{dG}{dn} d\sigma = 4\pi$, where the integration is extended to the whole of any surface S′ contained within S and containing within it the point M′ and the differentiation $\frac{d}{dn}$ is along the normal interior to the surface S′.

Since $G = H + 1/r$ is harmonic everywhere in T except at M′ where it is infinite, $\frac{dG}{dn}$ exists at every point upon S′. Enclose M′ by a small sphere Σ of radius ρ and centre M′. Now in the space between Σ and S′, G is harmonic, and hence $\int \frac{dG}{dn} d\sigma$ extended to the whole of the surfaces bounding that space is zero. [*vide*: Cor. 2 of § 2 in the article on the Problem of Dirichlet].

i.e.

$$\int_{S'} \frac{dG}{dn} d\sigma + \int_{\Sigma} \frac{dG}{dn} d\sigma = 0,$$

the differentiation $\frac{d}{dn}$ being along the normal interior to the space enclosed; in other words

$$\int_{S'} \frac{dG}{dn} d\sigma = \int_{\Sigma} \frac{dG}{dn} d\sigma,$$

the differentiation $\frac{d}{dn}$ being along the normal interior to the geometrical surface in both cases.

But

$$\int_{\Sigma} \frac{dG}{dn} d\sigma = \int_{\Sigma} \frac{d\left(\frac{1}{r}\right)}{dn} d\sigma.$$

$$= \int_{\Sigma} -\frac{1}{r^2} \frac{dr}{dn} d\sigma = \frac{1}{\rho^2} \int_{\Sigma} d\sigma = 4\pi.$$

For, the differentiation being along the normal interior to Σ, $\frac{dr}{dn} = -1$ in the case of the sphere.

$$\therefore \qquad \int_{S'} \frac{dG}{dn} d\sigma = 4\pi.$$

[*Note.*—This could be easily identified as a particular case of Gauss' theorem on the surface integral of normal force.]

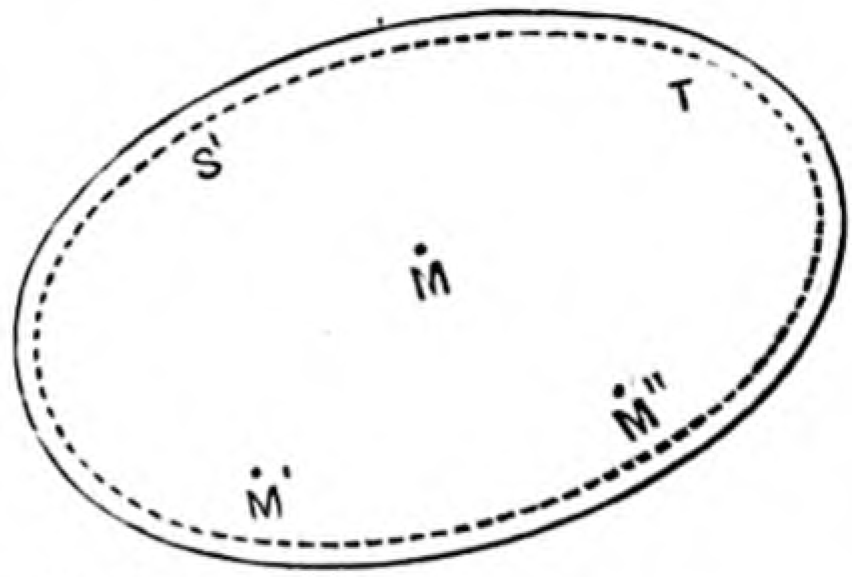
6°. Let T be a volume bounded by a surface S, and M′, M″ two points within it. Let G′ denote the function of Green relative to the

domain T and to the point M′, and G″ denote the function of Green relative to the same domain T and to the point M″. Then the value of G′ at M″ is equal to the value of G″ at M′.

That is to say, if we denote by G′ (M) and G″ (M) the values of the two functions G′ and G″ respectively at any point M, we shall have

$$G'(M'')=G''(M').$$

Let α' be a very small number and consider the surface S′ corresponding to $G'=\alpha'$. This surface S′ is very near to S and by taking α' sufficiently small, S′ can be made to contain within its interior both the point M′ and M″.



Now consider the integral

$$J=\int_{S'}\left(G'\frac{dG''}{dn}-G''\frac{dG'}{dn}\right)d\sigma$$

extended to the surface S′, where $\frac{d}{dn}$ is taken along the normal interior to S′.

This integral is well determinate since G′, G″, $\frac{dG'}{dn}$, $\frac{dG''}{dn}$ exist at all points of S′.

Around the two points M′ and M″ as centres, describe small spheres Σ′, Σ″ of radii ρ', ρ'' respectively, and consider the space contained between the surface S′ and the spheres Σ′, Σ″. In this space, G′ and G″ are harmonic, and hence, proceeding as in 5° above, we have

$$\int_{S'}\left(G'\frac{dG''}{dn}-G''\frac{dG'}{dn}\right)d\sigma=\int_{\Sigma'}\left(G'\frac{dG''}{dn}-G''\frac{dG'}{dn}\right)d\sigma$$
$$+\int_{\Sigma''}\left(G'\frac{dG''}{dn}-G''\frac{dG'}{dn}\right)d\sigma$$

where $\frac{d}{dn}$ is to be taken along the normals interior to the geometric surfaces.

Now, let M be the greatest value of G′ upon Σ'; then

$$\int_{\Sigma'} G'\frac{dG''}{dn}d\sigma < M\left|\int_{\Sigma'}\frac{dG''}{dn}d\sigma\right| = 0$$

because G″ is harmonic in Σ.

Next, consider the integral $\int_{\Sigma'} G''\frac{dG'}{dn}d\sigma$.

G″ is finite and continuous at M′. Therefore, on the sphere Σ', $G'' = G''(M') + \epsilon$ where ϵ vanishes with ρ'.

Thus $\int_{\Sigma'} G''\frac{dG'}{dn}d\sigma$ has for limiting value

$$G''(M')\int_{\Sigma'}\frac{dG'}{dn}d\sigma = 4\pi G''(M'), \text{ [by } 5^\circ].$$

Hence the limiting value of

$$\int_{\Sigma'}\left(G'\frac{dG''}{dn} - G''\frac{dG'}{dn}\right)d\sigma = -4\pi G''(M').$$

Similarly, the limiting value of

$$\int_{\Sigma''}\left(G'\frac{dG''}{dn} - G''\frac{dG'}{dn}\right)d\sigma = 4\pi G'(M'').$$

$$\therefore \qquad J = 4\pi\{G'(M'') - G''(M')\}.$$

Again $J = \int\left(G'\frac{dG''}{dn} - G''\frac{dG'}{dn}\right)d\sigma$ is the difference of two other integrals: *viz*

$$(1)\int_{S'} G'\frac{dG''}{dn}d\sigma; \quad (2)\int_{S'} G''\frac{dG'}{dn}d\sigma.$$

But $(1) = \alpha'\int_{S'}\frac{dG''}{dn}d\sigma$, because G′ is constant upon S′ and equals α'.

$= 4\pi\alpha'$ [by 5°.].

$\therefore$ (1) tends to zero with α'.

Nextly let α'' be the superior limit of G″ upon S′, so that we have

$$\left|\int_{S'} G''\frac{dG'}{dn}d\sigma\right| < \alpha''\int_{S'}\left|\frac{dG'}{dn}d\sigma\right|$$

$$< 4\pi\alpha'' \text{ [by } 5^\circ].$$

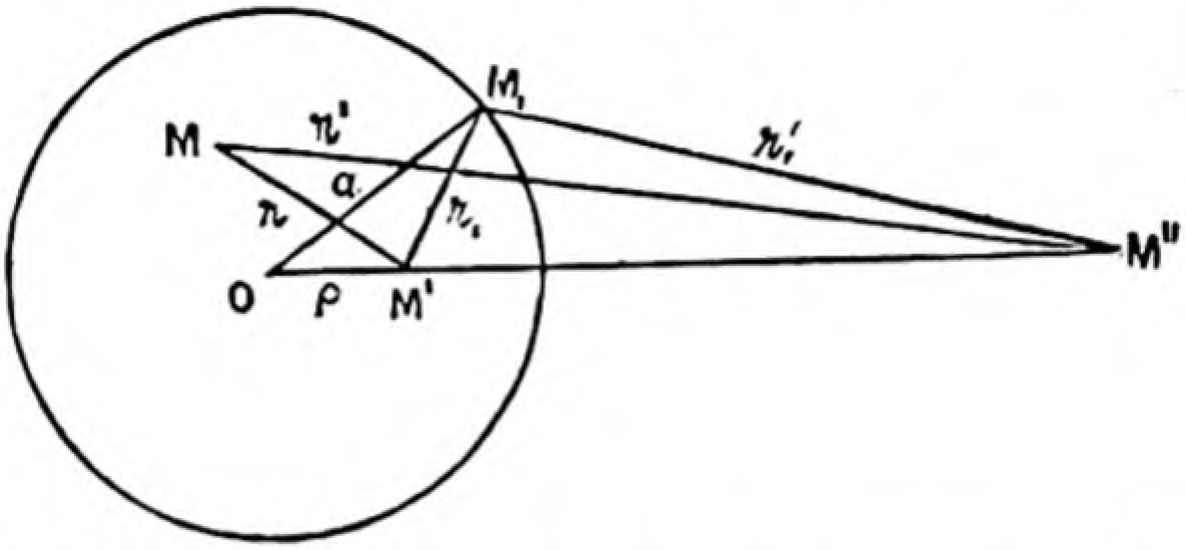
But α'' tends to zero simultaneously with α'; for, as α' tends to zero, the surface S′ tends to the surface S and hence α'' tends also to zero.

Thus J tends to zero simultaneously with α' and consequently the expression $4\pi\{G'(M'') - G''(M')\}$, which is equal to J, also tends to zero.

But this expression has a definite value; it is therefore necessarily zero; and we have

$$G'(M'')=G''(M').$$

6. *Value of Green's function in the case of a Sphere.* Let S be a sphere, O its centre and a its radius. Let M' be a point within the sphere. Let us calculate the function of Green relative to this point M'.



Let M'' be the point conjugate to M' with respect to the sphere, so that $OM'M''$ is a straight line and $OM'.OM''=a^2$.

Let M be any point within the sphere and M_1 a point on the sphere. Join OM_1, MM', MM'', M_1M', M_1M''; and put $MM'=r$, $MM''=r'$, $M_1M'=r_1$, $M_1M''=r_1'$ and $OM'=\rho$. Then $OM''=\frac{a^2}{\rho}$.; and by similar triangles we also have $\frac{r_1}{r_1'}=\frac{\rho}{a}$, whence

$$\frac{1}{r_1}=\frac{a}{\rho}.\frac{1}{r_1'} \quad \dots \quad \dots \quad \dots \quad (1)$$

Consider now $H=-\frac{a}{\rho}.\frac{1}{r'}$; H is harmonic within the sphere; for

$$\Delta\left(\frac{1}{r'}\right)=0.$$

Also, on the sphere $H=-\frac{a}{\rho}.\frac{1}{r_1'}=-\frac{1}{r_1}$, in virtue of the relation (1).

Hence $G=H+\frac{1}{r}=\frac{1}{r}-\frac{a}{\rho}.\frac{1}{r'}$, is the function of Green required.

Evidently, on the surface $G=\frac{1}{r_1}-\frac{a}{\rho}.\frac{1}{r_1'}=0$, by the relation (1).

Physical Interpretation: The above expression for G in the case of a sphere shows that the function of Green relative to a sphere and to

a given point within it is the total potential due to a unit charge of positive electricity placed at the given point and to a charge a/ρ of negative electricity at the conjugate point.

Note.—It could be easily shown that the property $G'(M'')=G''(M')$ in the case of a sphere, leads to Salmon's theorem on poles and polars.

7. *Comparison of the interior problems of Green and of Dirichlet*: The problem of Dirichlet has been given in the article on "The Problem of Dirichlet" quoted above. The following is the corresponding problem of Green.

"Given a volume T bounded by a surface S, calculate the function of Green relative to this volume T and to any one of the points in T."

Let U be the harmonic function in T which it is required to determine, in the problem of Dirichlet, from its values upon the surface S. The value U′ at a point M′ of T is given by the formula

$$4\pi\ U'=\int_S \overline{U}\frac{dG}{dn}d\sigma,$$

where G is the function of Green relative to the volume T and to the point M′, and $\overline{U}$ denotes the given value of U at the element $d\sigma$, and $\frac{d}{dn}$ is to be taken along the interior normal.

[This formula could be proved exactly like the formula $4\pi G''=\int G''\frac{dG}{dn}d\sigma$, proved in 6°, above.]

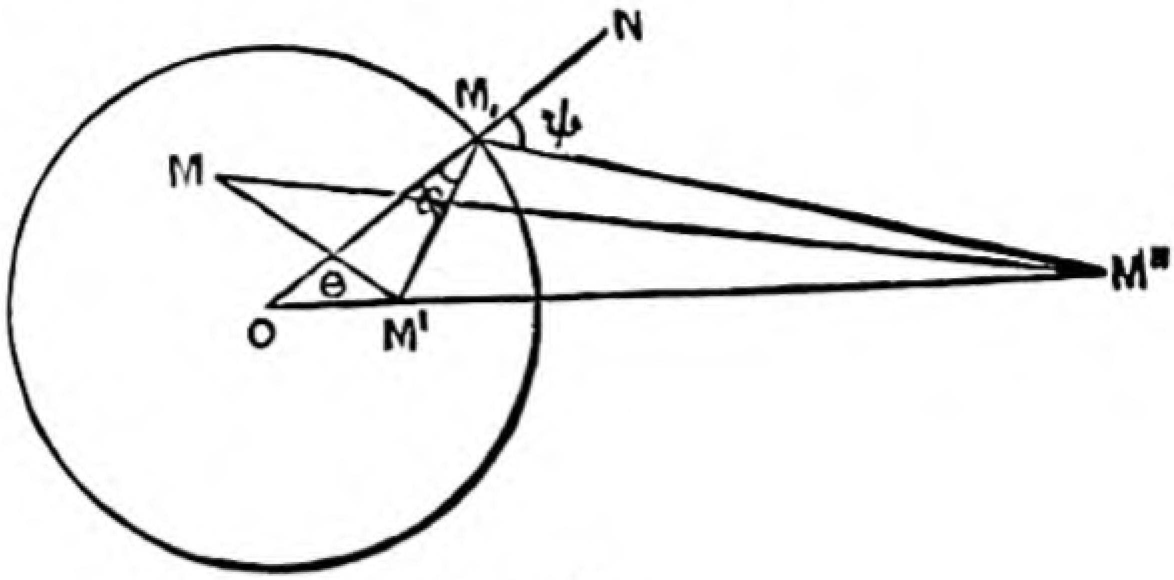
If, therefore, we know how to solve the problem of Green, we can calculate the value U′ of U at each point M′ of T, *i.e.*, we can solve the problem of Dirichlet.

Conversely, if we know how to solve the problem of Dirichlet, we can calculate the function H and consequently the function of Green.

The two problems of Green and of Dirichlet are, therefore, equivalent.

8. *Verification*: For verification take the case of a sphere for which we have solved both the problems independently.

With the notation already employed, we have



$$G=\frac{1}{r}-\frac{a}{\rho}\frac{1}{r'}$$

$$\frac{dG}{dn}=-\frac{1}{r^2}\frac{dr}{dn}+\frac{a}{\rho}\cdot\frac{1}{r'^2}\frac{dr'}{dn}$$

$$\frac{dr}{dn}=-\cos\phi,$$

$$\frac{dr'}{dn}=\cos\psi$$

where $\phi=M'M_1O$, $\psi=M''M_1\,N$;

$$\therefore \qquad \frac{dG}{dn}=\frac{\cos\phi}{r^2}+\frac{a}{\rho}\cdot\frac{\cos\psi}{r'^2}$$

$$=\frac{r\cos\phi}{r^3}+\frac{a}{\rho}\cdot\frac{r'\cos\psi}{r'^3}.$$

But $$r'^3=r^3\cdot\frac{a^3}{\rho^3}$$

$$\therefore \qquad \frac{dG}{dn}=\frac{r\cos\phi}{r^3}+\frac{\rho^2}{a^2}\frac{r'\cos\psi}{r^3}.$$

And $r\cos\phi=a-\rho\cos\theta$, where $\theta=M'OM_1$ and $r'\cos\psi=\dfrac{a^2}{\rho}\cos\theta-a.$

$$\therefore \qquad \frac{dG}{dn}=\frac{a-\rho\cos\theta}{r^3}+\frac{\rho^2}{a^2}\cdot\frac{\frac{a^2}{\rho}\cos\theta-a}{r^3}=\frac{a^2-\rho^2}{a\,r^3}.$$

Thus if U is the harmonic function sought in Dirichlet's Problem from its values upon the surface, we have

$$4\pi U=\int\overline{U}\,\frac{dG}{dn}d\sigma$$

$$=\int\frac{a^2-\rho^2}{a\,r^3}\,\overline{U}\,d\sigma$$

$$\therefore \qquad U=\frac{1}{4\pi a}\int\frac{a^2-\rho^2}{r^3}\,\overline{U}\,d\sigma.$$

which we know to be the solution.

SHORT NOTES.

On the Convergence of Infinite Power Chains.*

1. Let a_n be defined by the relation $a_n = a^{a_{n-1}}$, where n is a positive integer and a_0 denotes 1. Then obviously a_n stands for a simple chain of n links; we shall consider in this note the convergence of a_n as $n \to \infty$.

Suppose $a > 1$. It is obvious that a_n is an increasing function of n and hence it increases indefinitely or tends to a definite limit as $n \to \infty$. In the latter case, let $u = f(a)$ be the limit to which the infinite chain tends. Then u satisfies the equation.

$$a^u = u, \text{ or } a = u^{u^{-1}} \qquad \dots \qquad \dots \quad (1)$$

It is easily seen the $u^{u^{-1}}$ has a maximum value $k = e^{e^{-1}} = 1{\cdot}444$ nearly. Hence equation (1) has no real root if $a > k$. For such values a_n cannot converge, and so diverges.

If $a = k$, equation (1) gives $u = e$. In this case a_n may converge, in which case it will tend to the value e.

If $a < k$, there are real solutions of (1); and convergence is possible in this case also.

2. We shall now give a formal proof of the convergence of k_n when $n \to \infty$.

We have

$$k_1 = k = e^{e^{-1}};$$

$$k_2 = k^k = e^{e^{-\delta_1}}, \text{ where } \delta_1 = 1 - e^{-1} > o, < 1;$$

$$k_3 = k^{k_2} = e^{e^{-\delta_2}}, \text{ where } \delta_2 = 1 - e^{-\delta_1} > o, < 1;$$

and generally

$$k_n = k^{k_{n-1}} = e^{e^{-\delta_{n-1}}}, \text{ where } \delta_{n-1} = 1 - e^{-\delta_{n-2}} > o < 1.$$

Now consider the sequence $\delta_1, \delta_2, \delta_3, \dots\dots \delta_n$. From the relation

$$\delta_{r+1} = 1 - e^{-\delta_r}$$
$$= \delta_r - \frac{\delta_r^2}{2!} + \frac{\delta_r^3}{3!} - \dots\dots,$$

we infer $\delta_{r+1} < \delta_r$ if $\delta_r < 1$; so that $1 > \delta_1 > \delta_2 > \delta_3 \dots\dots > o$. The sequence is therefore a decreasing one, and so either vanishes

* Read before the Madras Presidency College Mathematical Association, 15-12-1915.

ultimately or tends to a limit $\delta<1$. In the latter case δ must satisfy the equation

$$\delta=\delta-\frac{\delta^2}{2!}+\frac{\delta^3}{3!}-\dots\dots$$

the only real root of which is $\delta=0$.

Hence

$$\operatorname*{Lt}_{n=\infty} k_n=e^{e^{-\delta}}=e.$$

Cor. If $1<a<k$, we have $a_n<k_n$.

$\therefore$ $\qquad\qquad \text{Lt } a_n<\text{Lt } k_n<e.$

Hence a_n converges for $a<k$.

4. Suppose $a<1$,

We have $\qquad a<a_n<1$, or $a_1<a_2<1$.

$\therefore$ $\qquad a^{a_1}>a^{a_2}>a$, since $a<1$.

i.e. $\qquad a_2>a_3>a_1$.

Similarly $\qquad a_3<a_4<a_2$;

and $\qquad a_4>a_5>a_3$.

Thus a_n oscillates, the amplitude gradually decreasing as n increases. Every even a is greater than every odd a. The even a's form a decreasing sequence and the odd a's an increasing one.

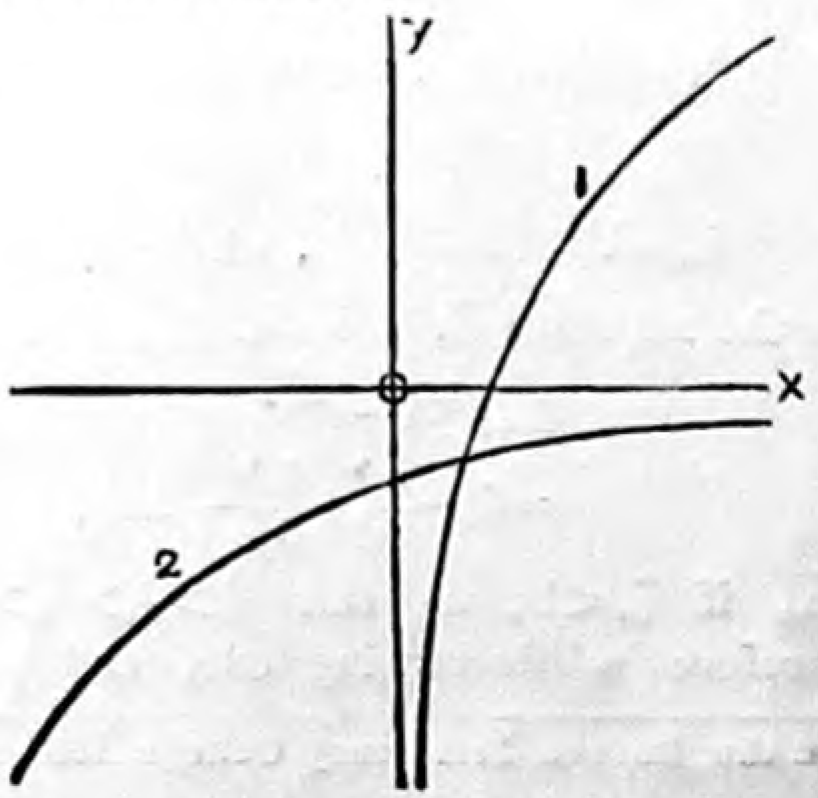
Let the limits of these sequences be λ and μ, respectively; so that λ, μ are real roots of the equation

$$a^{a^x}=x, \text{ or } a^x \log a=\log x,$$

and may be obtained as the points of intersection of the curves:

(1) $y=\log x$, and (2) $y=a^x \log a$, $(a<1)$.

The graphs of these curves are given below and it is obvious from the figure that there is only one real point of intersection. Hence $\lambda=\mu$, and a_n converges to a definite limit in much the same manner as the convergents of a continued fraction.



5. Let us next consider the general chain $a_1^{a_2^{a_3\cdots}}$ in which the a's are all greater than unity. The chain diverges if $a_r > a > k$, for all values of r beyond a particular number p, say.

For, the part of the infinite chain corresponding to such values is greater than $a^{a^{a\cdots}}\cdots$ whose divergence has been already proved.

(ii) If $a_r \overline{<} k$, after a certain value of r, we find that the part of the infinite chain beginning with a_r is less than k. Hence the whole chain converges.

A. Narasinga Rao.

Conical Envelope of a Conicoid.

1. Let $x^2/a^2+y^2/b^2+z^2/c^2=1$, be a conicoid, and (f, g, h) any point T. Also, let (ξ, η, ζ) be a point P on a tangent from T to the conicoid. Then, we have

$$\triangle\ \text{CPT} : \triangle\ \text{CDT} = \text{PT} : \text{CD}, \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

where CD is the semi-diameter parallel to TP.

$$\text{Also PT} : \text{CD} = (\xi-f) : al = (\eta-g) : bm = (\zeta-h) : cn$$

$$= [(\xi-f)^2/a^2+(\eta-g)^2/b^2+(\zeta-h)^2/c^2]^{\frac{1}{2}}, \quad \ldots \quad \text{(ii)}$$

D being denoted by (al, bm, cn).

From (i) and (ii), we find

$$\triangle\ \text{CPT} : \triangle\ \text{CDT} = [(\xi-f)^2/a^2+\ldots]^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Now $\quad 2.\triangle\ \text{CPT} = \text{CP.CT} \sin \text{PCT} = [(\xi g-\eta f)^2+\ldots]^{\frac{1}{2}}$

$\quad 2.\triangle\ \text{CDT} =$ product of semi-axes of the section CPT

$$= abc/(a^2l'^2+b^2m'^2+c^2n'^2)^{\frac{1}{2}},$$

where $l'x+m'y+n'z=o$ denotes the plane CPT, *viz*:

$$\begin{vmatrix} x & y & z \\ \xi & \eta & z \\ f & g & h \end{vmatrix} = 0.$$

Hence (iii) reduces to

$$[a^2(\xi g-\eta f)^2+\ldots\ldots] = [(\xi-f)^2/a^2+\ldots]a^2b^2c^2$$

$$= [b^2c^2(\xi-f)^2+\ldots\ldots], \quad \ldots \quad \ldots \quad \text{(iv)}$$

which is the equation to the conical envelope.

Cor :—Equation (iv) may be written in the form

$$[(\xi g-\eta f)^2+\ldots\ldots]=a^2[(\xi-f)^2+\ldots\ldots]$$

in the case of a sphere.

2. In the case of a conic $x^2/a^2+y^2/b^2=1$, the equation of the tangents from (f, g) is similarly written

$$(\xi g-\eta f)^2=b^2(\xi-f)^2+a^2(\eta-g)^2,$$

which reduces to

$$(\xi g-\eta f)^2=a^2[(\xi-f)^2+(\eta-g)^2]$$

for a circle.

M. T. Naraniengar.

Involution and (1, 1) Correspondence.

1. *Transformation of Point-pairs on a Line* :

For visualising certain relations between points on a line, which we term involution and (1, 1) correspondence, a convenient method may be employed. A line is one-dimensional if we regard it as composed of points, but is two-dimensional if regarded as made up of point-pairs. Thus every point-pair in a line can be placed in (1, 1) correspondence with a point in a plane. This is clearly seen to be possible if we specify the point-pair by two co-ordinates (p, q). It should be remarked however that since we are not supposed to make any distinction between the points composing the pair, (p, q) should be symmetric functions of the co-ordinates of the points. For example, if x_1, x_2, the distances of two points from a fixed point on the line be the roots of the equation

$$x^2+px+q=o,$$

then (p, q) can be taken to be the co-ordinates of the point-pair $(x_1\ x_2)$. (*vide* : Young : *Theory of Sets of Points*, Ch. VIII).

2. *Some Properties* :

Making this transformation of point-pairs in a line to points in a plane, we have

(1) Any continuous one-dimensional series of point-pairs corresponds to a curve.

(2) Any involution corresponds to a straight line. For, since an involution is completely determined by two point-pairs, the corresponding curve must be completely determined by two points.

(3) The locus in the plane of all repeated pairs such as (PP) in the line is a conic which shall be termed the fundamental conic.

For there are two such pairs in every involution. Hence the corresponding curve must cut every straight line in two points.

(4) If two point-pairs have a point in common, the line joining the corresponding points touches the fundamental conic.

For, the double points of the involution determined by two pairs having a common point P, coincide into the single point P. Hence in the plane the line representing the involution touches the fundamental conic.

(5) The point-pair formed by the double points of an involution is represented by the pole of the corresponding line *w. r. t.* the fundamental conic.

For, if the line cuts the conic in (p, q) corresponding to the pairs (PP), (QQ), the pair formed by the double points being (PQ) must, by the last theorem, lie on the tangents at both p and q.

Cor. Two point-pairs separating each other harmonically are represented by conjugate points *w. r. t.* the fundamental conic.

The reciprocal theorem concerning pole and polar is thus seen to follow from the reciprocity of the harmonic relation.

(6) Point-pairs consisting of points in (1, 1) correspondence are represented by a conic having double contact with the fundamental conic.

For, if P is any point, the correspondence will carry P into P_1 and some other point P_2 into P. Thus there are two pairs (PP_1) (PP_2) containing P which shows that any tangent to the fundamental conic cuts the corresponding locus in two points. This locus is therefore a conic.

Since a (1, 1) correspondence has two and only two double pairs, it follows that this conic must have double contact with the fundamental conic in the points corresponding to the double points.

R. VYTHYNATHASWAMY.

The Face of the Sky for March and April.

The Sun

enters the first point of Aries on March 21 at 4-30 A.M. and Taurus on April 20 at 3-30 P.M.

The Moon

	March.				*April.*			
	D.	H.	M.		D.	H.	M.	
New Moon	4	9	27	A.M.	2	9	51	P.M.
First Quarter ...	12	12	3	A.M.	10	7	65	P.M.
Full Moon ...	19	10	56	P.M.	18	10	37	A.M.
Last Quarter ...	26	9	52	P.M.	25	4	8	A.M.

The Planets.

Mercury attains its greatest elongation (27° 6′ West) on March 2. It is in superior conjunction on April 15 and in conjunction with the Moon on March 2 and April 2, with Jupiter on April 9 and with Uranus on March 5.

Venus continues an evening star. It attains its greatest elongation (45° 33′ East) on April 24. It is in conjunction with the Moon on March 7 at 6-30 P.M. and April 6 at 5-30 P.M. and with Arietis on March 27.

Mars which was in opposition on February 9—one of the most favourable oppositions—becomes stationary on March 21. It is in conjunction with the Moon on March 16 and on April 12 at 7-30 P.M.

Jupiter is in conjunction with the Sun on April 1 at 7-30 P.M., and with the Moon on March 6 and April 3 and April 30.

Saturn is stationary on March 11. It is in quadrature on March 31. It is in conjunction with the Moon on March 13 and on April 9.

Uranus is in conjunction with the Moon on March 2, March 29 and April 26.

Neptune is stationary on April 10. It is in quadrature on April 20. It is in conjunction with the Moon on March 15 and on April 11.

N.B.—The total eclipse of the Sun on February 3 is invisible in India.

V. RAMESAM.

SOLUTIONS.

Question 541.

(S. Ramanujan) :—Prove that

$$1+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\frac{1}{1\cdot 3\cdot 5\cdot 7}+\dots+\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\,\frac{5}{1+\dots}=\sqrt{\frac{\pi e}{2}}.$$

Remarks by K. B. Madhava.

We have Prym's identity (Bromwich, p. 9, 3 and 17)

$$\frac{1}{a}+\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}+\frac{x^3}{a(a+1)(a+2)(a+3)}+\dots\dots$$

$$=e^x\left[\frac{1}{a}-\frac{x}{1!}\,\frac{1}{a+1}+\frac{x^2}{2!}\,\frac{1}{a+2}-\frac{x^3}{3!}\,\frac{1}{a+3}+\dots\dots\right]$$

$$=\frac{e^x}{x^a}\int_0^x e^{-x}x^{a-1}\,dx.$$

Putting $a=x=\frac{1}{2}$, we find

$$2\left\{1+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\frac{1}{1\cdot 3\cdot 5\cdot 7}+\dots\infty\right\}=\sqrt{2e}\int_0^{\frac{1}{2}} e^{-x}x^{-\frac{1}{2}}\,dx.$$

$$=2\sqrt{2e}\int_0^{\frac{1}{\sqrt{2}}} e^{-t^2}dt, \text{ putting } x=t^2;$$

so that

$$1+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\frac{1}{1\cdot 3\cdot 5\cdot 7}+\dots=\sqrt{2e}\int_0^{\frac{1}{\sqrt{2}}} e^{-t^2}\,dt. \qquad \dots \quad (1)$$

Now consider the continued fraction

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\,\frac{5}{1+\dots\dots} \qquad (A)$$

which we know to be convergent (Chrystal, p. 525, Ex. 7); this can be transformed in various ways.

The three following methods are given in Oskar Perron "*Die Lehre von den Kettenbruchen.*" (Teubner, 1913).

(i) The first (pp. 294—298) gives at once the desired result of this example.

$$\text{Let } \phi(\alpha,\beta)=\frac{1}{\Gamma(\beta)}\int_0^\infty \frac{e^{-u}u^{\beta-1}}{(1+xu)^\alpha}du \quad \dots \qquad \dots \quad (2)$$

where x is real and positive and $\beta>o$; but when $\beta=0$, let $\phi(\alpha, o)=1$. (3)

Integrating (2) by parts

$$\phi(\alpha,\beta)=\phi(\alpha,\beta+1)+\alpha x\phi(\alpha+1,\beta+1),$$

and a set of results follows successively, from which we get (as in Chrystal, Ch. 34, §§ 20—22)

$$\frac{\phi(\alpha,\beta)}{\phi(\alpha,\beta+1)}=1+\frac{\alpha x}{1+}\;\frac{(\beta+1)x}{1+}\;\frac{(\alpha+1)x}{1+}\;\frac{(\beta+2)x}{1+\dots}\quad\dots\quad(4)$$

which is true except for $\alpha=0$ or any negative integer.

Putting $\beta=0$ and inverting, we have with the aid of (3),

$$\frac{1}{1+}\;\frac{\alpha x}{1+}\;\frac{1\cdot x}{1+}\;\frac{(\alpha+1)x}{1+}\;\frac{2\cdot x}{1+}\dots=\phi(\alpha,1)$$

$$=\int_0^\infty e^{-u}(1+xu)^{-\alpha}\,du.\qquad(5)$$

Now put $1+xu=xv$ in this last integral and then $x=\frac{1}{z}$ we have;

$$z^{\alpha-1}e^{z}\int_z^{\infty} e^{-v}v^{-\alpha}\,dv=\frac{1}{z+}\;\frac{\alpha}{1+}\;\frac{1}{z+}\;\frac{\alpha+1}{1+}\;\frac{2}{z+}\;\frac{\alpha+2}{z+}\;\frac{3}{z+\dots}\qquad(6)$$

wherein the only conditions are $z>o$ and α is real.

This is a useful result and is also given in Legendre: ***Fonctions Elliptiques***: Tome ii, Ch. 17.

Now in this first put

$$\alpha=\tfrac{1}{2},\ v=t^2;\ z=\xi^2,$$

and multiply the left hand side of (5) by ξ^2.

We have

$$2\xi\,e^{\xi^2}\int_{\xi}^{\infty} e^{-t^2}\,dt=\frac{1}{1+}\;\frac{\frac{1}{2\xi^2}}{1+}\;\frac{\frac{2}{2\xi^2}}{1+}\;\frac{\frac{2}{2\xi^2}}{1+\dots}$$

This result is also given in Laplace, *Celestial Mechanics*, vol. IV, p. 257; and in Jacobi: *Ges Werke*, Bd. VI, pp. 76—78.

If we simply put $\xi^2=\frac{1}{2}$, we get

$$\frac{1}{1+}\;\frac{1}{1+}\;\frac{2}{1+}\;\frac{3}{1+}\dots\dots=\sqrt{2e}\int_{\sqrt{\frac{1}{2}}}^{\infty} e^{-t^2}\,dt.\ \dots\quad\dots\quad(7)$$

Combining (1) and (7) we have

$$1+\frac{1}{1\cdot3}+\frac{1}{1\cdot3\cdot5}+\dots\dots+\frac{1}{1+}\;\frac{1}{1+}\;\frac{2}{1+}\;\frac{3}{1+}\;\frac{4}{1+}\dots\dots$$

$$=\sqrt{2e}\int_0^\infty e^{-t^2}\,dt=\sqrt{2e}.\frac{\sqrt{\pi}}{2}=\sqrt{\frac{\pi e}{2}}.$$

(2) In the second method referred to, M. Perron (pp. 380—392) obtains the same result by taking the corresponding 'modified'

asymptotic series and applies to this the general method for expressing asymptotic series in the form of a continued fraction.

Bromwich (p. 267 and §§ 136—139) has also the same discussion; he shows that the series

$$x-1!\,x^2+2!\,x^3-3!\,x^4+\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (8)$$

which formally satisfies the differential equation

$$x^2\frac{dy}{dx}+y=x \qquad \ldots \qquad \ldots \qquad \ldots \qquad (9)$$

has the the definite integral solution

$$y=\int_0^\infty \frac{xe^{-tx}}{1+xt}\,dt, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (10)$$

and the solution in the form of a continued fraction

$$y=\frac{x}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\,\frac{3x}{1+}\ldots \qquad \ldots \qquad \ldots \qquad (11)$$

which is the same as (6), with $\alpha=1$ and $z=1$.

(3) The third method of M. Perron (pp. 469—472) is to be deduced from the general result of expressing a function in continued fraction if it satisfies a differential equation: *viz.*,

If

$$y=Q_0y'+P_1y'' \qquad \ldots \qquad \ldots \qquad \ldots \qquad (12)$$

since the n^{th} differential of this is

$$y^{(n)}=Q_n\,y^{(n+1)}+P_{n+1}\,y^{(n+2)} \qquad \ldots \qquad \ldots \qquad (13)$$

where

$$Q_n=\frac{Q_{n-1}+P_n'}{1-Q'_{n-1}},\ \text{and}\ P_{n+1}=\frac{P_n}{1-Q'_{n-1}}, \qquad \ldots \qquad \ldots \qquad (14)$$

we easily see that

$$\frac{y}{y'}=Q_0+\frac{P_1}{Q_1+}\,\frac{P_2}{Q_2+}\,\frac{P_3}{Q_3+}\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (15)$$

where the P's and Q's are determined as in (14).

If we adopt this method for the function

$$y=\int_0^\infty e^{ux-\frac{1}{2}u^2}\,u^{\alpha-1}\,du \qquad \ldots \qquad \ldots \qquad (16)$$

by partial integration we have

$$y=\frac{1}{\alpha}\int_0^\infty e^{ux-\frac{1}{2}u^2}\,d(u^\alpha)$$

$$=-\frac{1}{\alpha}\int_0^\infty e^{ux-\frac{1}{2}u^2}(x-u)\,u^\alpha\,du$$

$$=-\frac{x}{\alpha}y'+\frac{1}{\alpha}y'', \qquad \ldots \qquad \ldots \qquad \ldots \qquad (17)$$

and hence differentiating (17)

$$y' = -\frac{1}{\alpha}y' - \frac{x}{\alpha}y'' + \frac{1}{\alpha}y'',$$

i.e.

$$y' = \frac{-x}{\alpha+1}y'' + \frac{1}{\alpha+1}y''' :$$

or generally

$$y^{(n)} = \frac{-x\,y^{(n+1)}}{\alpha+n} + \frac{1}{\alpha+n}y^{(n+2)}. \qquad \cdots \quad (18)$$

Therefore from the general formula (15)

$$\frac{y}{y'} = \frac{-x}{\alpha} + \cfrac{\frac{1}{\alpha}}{\cfrac{-x}{\alpha+1} + \cfrac{\frac{1}{\alpha+1}}{\cfrac{-x}{\alpha+2} + \cdots\cdots}}$$

which can be transformed into

$$\frac{-x}{\alpha-}\;\frac{1+1/\alpha}{-\alpha+}\;\frac{\alpha+2}{-x+}\;\frac{\alpha+3}{-x+} \quad \cdots \quad \cdots \quad \cdots \quad (19)$$

by multiplying by $\alpha+1$ after the second term.

If in this we put $x = -\xi$ and multiply by α, we have

$$\xi + \frac{\alpha+1}{\xi+}\;\frac{\alpha+2}{\xi+}\;\frac{\alpha+3}{\xi+}\cdots = \alpha\,\frac{\int_0^\infty e^{-u\xi-\frac{1}{2}u^2}u^{\alpha-1}du}{\int_0^\infty e^{-u\xi-\frac{1}{2}u^2}u^{\alpha}du} \quad \cdots \quad (20)$$

If in this again, we put $\alpha = \xi = 1$, we fall back on (A)

$$1 + \frac{2}{1+}\;\frac{3}{1+}\;\frac{4}{1+\ldots} = \frac{\int_0^\infty e^{-u-\frac{1}{2}u^2}du}{\int_0^\infty e^{-u-\frac{1}{2}u^2}u\,du} \quad \cdots \quad \cdots \quad (21)$$

which are familiar integrals.

Question 593.

(S. Narayana Aiyangar, M.A.) :—Shew that

$$\frac{\frac{1}{a} + \frac{x}{a+1} + \frac{x^2}{1\cdot 2}\cdot\frac{1}{a+2} + \ldots\ldots\text{to }\infty}{\frac{1}{a+1} + \frac{x}{a+2} + \frac{x^2}{1\cdot 2}\,\frac{1}{a+3} + \ldots\ldots\text{to }\infty}$$

is equal to

$$\frac{\frac{1}{a} - \frac{x}{a(a+1)} + \frac{x^2}{a(a+1)(a+2)} - \ldots\text{to }\infty}{\frac{1}{1+a} - \frac{x}{(a+1)(a+2)} + \frac{x^2}{(a+1)(a+2)(a+3)} - \ldots\text{to }\infty}.$$

Additional Solution by C. Krishnamachary.

This can be easily solved by the application of the method of finite differences. We have the equality of operators

$$E = 1 + \Delta.$$

Hence

$$1 + xE + \frac{x^2E^2}{2\,!} + \ldots = 1 + x(1+\Delta) + \frac{x^2}{2!}(1^2+\Delta)^2 + \ldots$$

$$= (1 + x + \frac{x^2}{2!} + \ldots)\ (1 + x\Delta + x^2\frac{\Delta^2}{2\,!} + \ldots);$$

since if $E\,(x) = 1 + x + \frac{x^2}{2\,!} + \cdots$,

$$E(x) \times E(y) = E(x+y)]$$

$$= e^x(1 + x\Delta + \frac{x^2}{2!}.\Delta^2 + \ldots).$$

Operate upon the function $\frac{1}{a}$. Then

$$(1 + xE + \frac{x^2}{2!}E^2 + \ldots)\frac{1}{a} = \frac{1}{a} + xE\ \left(\frac{1}{a}\right) + \frac{x^2}{2!}E^2\left(\frac{1}{a}\right) + \ldots$$

$$= \frac{1}{a} + \frac{x}{a+1} + \frac{x^2}{2\,!}\cdot\frac{1}{a+2} + \frac{x^3}{3\,!}\cdot\frac{1}{a+3} + \ldots$$

and

$$e^x(1 + x\Delta + \frac{x^2}{2\,!}.\Delta^2 + \ldots)\frac{1}{a} = e^x.\left(\frac{1}{a} + x.\Delta\left(\frac{1}{a}\right) + \frac{x^2}{2!}.\Delta^2\left(\frac{1}{a}\right) + \ldots\right)$$

$$= e^x\left(\frac{1}{a} - \frac{x}{a(a+1)} + \frac{x^2}{a(a+1)(a+2)} - \ldots\right);$$

since $\Delta\left(\frac{1}{a}\right) = (E-1)\frac{1}{a} = \frac{1}{a+1} - \frac{1}{a} = -\frac{1}{a(a+1)}$.

$$\Delta^2\left(\frac{1}{a}\right) = \Delta\left(\Delta\ \left(\frac{1}{a}\right)\right) = \frac{|\underline{2}}{a(a+1)(a+2)};$$ and so on].

Hence we have the equality of the two series

$$\frac{1}{a} + \frac{x}{a+1} + \frac{x^2}{2!}\cdot\frac{1}{a+2} + \frac{x^3}{3!}\cdot\frac{1}{a+3} + \ldots$$

$$= e^x\left\{\frac{1}{a} - \frac{x}{a(a+1)} + \frac{x^2}{a(a(+1)(a+2)} + \ldots\right\} \quad \ldots \quad (1)$$

Put $a+1$ for a, and we similarly have

$$\frac{1}{a+1} + \frac{x}{a+2} + \frac{x^2}{2!}\cdot\frac{1}{a+3} + \ldots$$

$$= e^x\left\{\frac{a+1}{1} - \frac{x}{(a+1)(a+2)} + \frac{x^2}{(a+1)(a+2)(a+3)}\right\} \quad \ldots \quad (2)$$

Hence from (1) and (2) we have

$$\frac{\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{2!}\cdot\frac{1}{a+2}+\dots}{\frac{1}{a+1}+\frac{x}{a+2}+\frac{x^2}{2!}\cdot\frac{1}{a+3}+\dots}=\frac{\frac{1}{a}-\frac{x}{a(a+1)}+\dots}{\frac{1}{a+1}-\frac{x}{(a+1)(a+2)}+\dots}.$$

Question 617

(S. Narayana Aiyar, M.A.) :—If $F(\alpha, \beta, \gamma, x)$ denote the hypergeometric series $1+\frac{\alpha.\beta}{\gamma.1}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1).\quad 1.2}x^2+\dots$, show that

$$\sum_{r=\infty}^{r=0}\left\{\frac{P(b)P(\gamma)P(a-b+1)P(\alpha+r)P(\beta+r)}{P(\alpha)P(\beta)P(b+r)P(\gamma+r)P(a-b-r+1)}\cdot\frac{x^r}{P(r+1)}.F(\alpha',\beta',\gamma',x)\right\}$$

$$=1+\frac{a.\alpha.\beta}{b.\gamma.1}x+\frac{a(a+1).\alpha(\alpha+1).\beta(\beta+1)}{b(b+1).\gamma(\gamma+1).\quad 1.2}x^2+\dots$$

where P stands for Γ, and

$$\alpha'=\alpha+r\,;\ \beta'=\beta+r\,;\ \gamma'=\gamma+r.$$

Examine the case when $\alpha=\gamma$.

Solution by K. B. Madhava.

By comparing the co-efficients of x^n, we have to show that

$$\frac{P(\alpha+n)P(\beta+n)P(a+n)P(b)P(\gamma)}{P(\alpha)P(\beta)P(a)P(b+n)P(\gamma+n)P(n+1)}$$

is equal to the series

$$\sum_{r=0}^{n}\frac{P(\alpha+n)P(\beta+n)P(\gamma+r)}{P(\alpha+r)P(\beta+r)P(\gamma+n)P(n-r+1)}\times$$
$$\frac{P(b)P(\gamma)P(a-b+1)P(\alpha+r)P(\beta+r)}{P(\alpha)P(\beta)P(b+r)P(\gamma+r)P(a-b-r+1)P(r+1)}$$

i.e. $$\sum_{r=0}^{n}\frac{P(\alpha+n)P(\beta+n)P(b)P(\gamma)P(a-b+1)}{P(\gamma+n)P(\alpha)P(\beta)P(b+r)P(a-b-r+1)P(r+1)P(n-r+1)};$$

hence it is required to show that

$$\frac{P(a+n)}{P(a)P(b+n)P(n+1)}=\sum_{0}^{n}\frac{P(a-b+1)}{P(b+r)P(a-b-r+1).P(r+1)P(n-r+1)}.$$

Multiplying both sides by $P(b)P(n+1)$, we have to show

$$1+\frac{(a-b)}{b}\cdot\frac{n}{1}+\frac{(a-b)(a-b-1)}{b(b+1)}\cdot\frac{n(n-1)}{1.2}+\dots\text{up to }(n+1)\text{ terms}$$

$$=\frac{(a+n-1)(a+n-2)\dots(a+1)a}{(b+n-1)(b+n-2)\dots(b+1)b},$$

and this has been done in the solution to Q. 616 (J. I. M. S., Vol. VII, p. 195); putting $b-a=k$ and $b=l$, we get, as required to be shown,

$$1-\frac{k}{l}\cdot\frac{n}{1}+\frac{k(k+1)}{l(l+1)}\cdot\frac{n(n-1)}{1.2}-\ldots \text{ up to } (n+1) \text{ terms}$$
$$=\frac{(l-k)(l-k+1)\ldots(l-k+n-1)}{l(l+1)\ldots(l+n-1)}.$$

When $\alpha=\gamma$, the right-hand side of the given series reduces to $F(a,\beta,b,x)$, while in the left-hand side $F(\alpha',\beta',\gamma',x)$ reduces to

$$(1-x)^{-\beta-r};$$

and therefore we get

$$F(a,\beta,b,x)=\sum_0^\infty\left\{\frac{P(b)P(a-b+1)P(\beta+r)}{P(\beta)P(b+r)P(a-b-r+1)}\frac{x^r}{P(r+1)}(1-x)^{-\beta-r}\right\}$$

which, after some manipulations, is seen to be

$$=(1-x)^{-\beta}.F\left(\beta,b-a,b,\frac{x}{x-1}\right)$$

being one of the 23 other forms in which F can be expressed.

Question 627.

(K. V. Anantanarayana Sastri, B.A.) :—Four spheres of radii a,b,c,d intersect at right angles. Show that the volume of the tetrahedron formed by their centres is

$$\frac{1}{6}\,a\,b\,c\,d\,(a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}.$$

Additional Solutions (1) *by K. B. Madhava and* (2) *by R. Vythynathaswamy*

(1) Take for the equations of the four spheres

$$x^2+y^2+z^2-a^2=0,$$

and $\quad x^2+y^2+z^2+2x_rx+2y_ry+2z_rz-k_r=0\ (r=1,2,3).$

By the conditions of the problem

$$x_r^2+y_r^2+z_r^2-k_r=b^2,\ c^2 \text{ or } d^2 \text{ according as } r \text{ is } 1, 2, 3.$$

and $$k_r=a^2$$

Let $\quad (\lambda\mu)\equiv x_\lambda x_\mu+y_\lambda y_\mu+z_\lambda z_\mu=\frac{1}{2}\left(k_\lambda+k_\mu\right)=a^2,$

where λ is any of the quantities 1, 2, 3 and μ either of the other. The volume of the tetrahedron formed by the centres

$$=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=\frac{1}{6}\begin{vmatrix} \Sigma x_1^2 & (12) & (13) \\ (12) & \Sigma x_2^2 & (23) \\ (13) & (23) & \Sigma x_3^2 \end{vmatrix}^{\frac{1}{2}}$$

$$=\frac{1}{6}\begin{vmatrix} a^2+b^2 & a^2 & a^2 \\ a^2 & a+c^2 & a^2 \\ a^2 & a^2 & a^2+d^2 \end{vmatrix}^{\frac{1}{2}}=\frac{1}{6}a^3\begin{vmatrix} 1+b^2/a^2 & 1 & 1 \\ 1 & 1+c^2/a^2 & 1 \\ 1 & 1 & 1+d^2/a^2 \end{vmatrix}^{\frac{1}{2}}$$

$$=\frac{1}{6}b\,c\,d\left(1+\frac{a^2}{b^2}+\frac{a^2}{c^2}+\frac{a^2}{d^2}\right)^{\frac{1}{2}}$$

$=\frac{1}{6}\,a\,b\,c\,d\left(a^{-2}+b^{-2}+c^{-2}+d^{-2}\right)^{\frac{1}{2}}$. (Burnside and Panton, p. 299, Ex. 20)

This is set as an exercise in Aldis's *Solid Geometry*, p. 137.

(2) The volume of a tetrahedron in terms of the edges is given by

$$144V^2=\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (12)^2 & (13)^2 & (14)^2 \\ 1 & (21)^2 & 0 & (23)^2 & (24)^2 \\ 1 & (31)^2 & (32)^2 & 0 & (34)^2 \\ 1 & (41)^2 & (42)^2 & (43)^2 & 0 \end{vmatrix}$$

This is got at once by multiplying the determinants

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+x_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y^4 & z_4 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -2x_1 & -2y_1 & -2z_1 & x_1^2+y_1^2+z_1^2 \\ 1 & -2x_2 & -2y_2 & -2z_2 & x_2^2+y_2^2+z_2^2 \\ 1 & -2x_3 & -2y_3 & -2z_3 & x_3^2+y_3^2+z_3^2 \\ 1 & -2x_4 & -2y_4 & -2x_4 & u_4^2+y_4^2+z_4^2 \end{vmatrix}$$

Putting in the above expression $(12)^2=a^2+b^2$, $(13)^2=a^2+c^2$, etc., the resulting determinant $=8\Sigma(b^2c^2d^2)$. (See Question 657.)

Hence $V=\frac{1}{6}abcd(a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}$.

Question 629.

(S. Ramanujan) :—Prove that :

$$\frac{\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2 x}\cos\left(\pi n^2\sqrt{1-x^2}\right)}{\sum_{n=1}^{\infty} e^{-\pi n^2 x}\sin\left(\pi n^2\sqrt{1-x^2}\right)}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}$$

and deduce the following :

$$(a)\quad \frac{\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2}}{\frac{1}{2}+\sum_{n=1}^{\infty} e^{-5\pi n^2}}=\sqrt{(5\sqrt{5}-10)}.$$

$$(b)\quad \sum_{1}^{\infty} e^{-\pi n^2}\left(\pi n^2-\frac{1}{4}\right)=\frac{1}{8}.$$

Solution (1) *by K. B. Madhava,* (2) *by N. Durai Rajan,* (3) *by M. Bhimasena Rao.*

(1) The denominator and the second term of the numerator in the first problem are respectively given by the imaginary and the real parts of the integral

$$\int_0^{\infty} e^{-\pi(x-i\sqrt{1-x^2})t^2}dt$$

the value of which $=\dfrac{\sqrt{\pi}}{2}\cdot\dfrac{1}{\sqrt{\pi(x-i\sqrt{1-x^2})}}=\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{(x-i\sqrt{1-x^2})}}$.

To separate the real and imaginary parts of this, put $x=\cos\psi$; and it is easy to see that the expression is equal to

$$\frac{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+x}{2}}}{\frac{1}{2}\sqrt{\frac{1-x}{2}}}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}.$$

Now, for (*b*) $\displaystyle\sum_{1}^{\infty} \pi n^2 e^{-\pi n^2}=\pi\int_0^{\infty} t^2 e^{-\pi t}\,dt$

$$=\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-t}.t^{\frac{3}{2}}dt=\frac{1}{4}$$

$$-\frac{1}{4}\sum_{8}^{\infty} e^{-\pi n^2}=-\frac{1}{4}\int_0^{\infty} e^{-\pi t^2}dt=-\frac{1}{4}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=-\frac{1}{8};$$

whence result (*b*) follows.

(2) Let

$$C=1+2\sum_{n=1}^{n=\infty} e^{-\pi n^2 x}\cos\left(\pi n^2\sqrt{1-x^2}\right)$$

$$S=\quad 2\sum e^{-\pi n^2 x}\sin\left(\pi n^2\sqrt{1-x^2}\right).$$

Substitute $x=\cos\theta$;

$$C+iS=1+2\sum e^{-\pi n^2\cos\theta}e^{\pi n^2 . i\sin\theta}$$

$$=1+2\sum e^{-\pi n^2 e^{-i\theta}} \quad \dots \quad \dots \quad \dots \quad (1)$$

$$C-iS=1+2\sum e^{-\pi n^2 e^{-\theta}} \quad \dots \quad \dots \quad \dots \quad (2)$$

Let
$$q=e^{-\pi\frac{K'}{K}}=e^{-\pi e^{i\theta}}.$$

$$\therefore \quad \frac{K'}{K}=e^{i\theta}=\frac{\psi(\sqrt{1-k})}{\psi(k)} \text{ say.}$$

$$\therefore \quad e^{-i\theta}=\frac{\psi(k)}{\psi(\sqrt{1-k^2})}=\frac{\psi(\sqrt{1-k'^2}}{\psi(k')}, \text{where } k^2+k'^2=1.$$

Let
$$q_1=e^{-\pi\frac{\Delta'}{\Delta}}=e^{-\pi e^{-i\theta}}.$$

$$\therefore \quad e^{-i\theta}=\frac{\Delta'}{\Delta}.$$

We see that $\Delta'=K$ and $\Delta=K'$.

Now
$$C-iS=1+2q+2q^4+2q^9+\dots$$
$$=\sqrt{\frac{K}{\frac{1}{2}\pi}}. \quad \text{(See, Greenhill's } \textit{Elliptic Fns}\text{)}$$

and
$$C+iS=\sqrt{\frac{\Delta}{\frac{1}{2}\pi}}.$$

$$\therefore \quad \left(\frac{C+iS}{C+iS'}\right)^2=\frac{\Delta}{K}=\frac{K'}{K}=e^{i\theta}.$$

$$\therefore \quad \frac{C+iS}{C+iS}=e^{\frac{1}{2}\theta i}=\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta.$$

Multiplying and equating real and imaginary parts, we get

$$(1+\cos\tfrac{1}{2}\theta)S=C\sin\tfrac{1}{2}\theta;\ C(1-\cos\tfrac{1}{2}\theta)=S.\sin\tfrac{1}{2}\theta.$$

$$\therefore C.\sin\frac{\theta}{4}=S\cos.\frac{\theta}{4}$$

$$\therefore \quad \frac{C}{S}=\frac{\cos\frac{\theta}{4}}{\sin\frac{\theta}{4}}=\cot\frac{\theta}{4}.$$

Replacing $x=\cos\theta$, we get the expression given in the question.

$$(b) \quad \sum_{n=1}^{n=\infty} e^{-\pi n^2}\left(\pi n^2-\frac{1}{4}\right)=\tfrac{1}{8}.$$

Let S be equal to the left hand side and, $e^{-\pi}=q$,

$$\text{so that } \frac{K'}{K}=1 \text{ and } k=k'=\frac{1}{\sqrt{2}}.$$

$$\therefore\ S=\Sigma q^{n^2}(\pi n^2-\tfrac{1}{4})$$

$$=\pi(q+4q^4+9q^9+16\,q^{16}+\ldots)-\tfrac{1}{4}(q+q^4+q^9+\ldots)$$

Now $\quad 1+2q+2q^4+\ldots=\sqrt{\frac{K}{\frac{1}{2}\pi}}.$

Differentiating with respect to q and remembering that

$$\log q=-\pi K'/K$$

$$2+8\,q^3+18\,q^8+\ldots=\tfrac{1}{2}\left(\frac{1}{\pi K}\right)^{\frac{1}{2}}\frac{dK}{dq}$$

$$\therefore\ 2(q+4q^4+9q^9+\ldots)=\tfrac{1}{2}\left(\frac{1}{\frac{1}{2}\pi K}\right)^{\frac{1}{2}}\frac{q\,dK}{dq}.)$$

Hence S = Limit of

$$\pi.\tfrac{1}{4}\left(\frac{1}{\frac{1}{2}\pi K}\right)^{\frac{1}{2}}q.\frac{dK}{dq}-\frac{1}{4}\left(\tfrac{1}{2}\sqrt{\frac{K}{\frac{1}{2}\pi}}-\tfrac{1}{2}\right)$$

$$=\frac{1}{8}+\mathrm{Lt}\frac{1}{4}\sqrt{\frac{1}{\frac{1}{2}\pi}}\left[\pi.K^{-\frac{1}{2}}.\ q\frac{dK}{dq}-\tfrac{1}{2}K^{\frac{1}{2}}\right]$$

$\therefore$ The expression within the square brackets is seen to be zero,

(3) We know that $\int_0^\infty e^{-t^2}2\cos 2\,xt\,dt=\sqrt{\pi}e^{-x^2}.$

$$\therefore\ \int_0^\infty e^{-t^2}\left(1+2\cos 2\,xt+2\cos 4\,xt+\ldots\ldots+2\cos 2nxt\right)dt$$

$$=\sqrt{\pi}\left\{\frac{1}{2}+\sum_1^n e^{-n^2x^2}\right\}.$$

$$\text{The left hand side}=\int_0^\infty e^{-t^2}.\frac{\sin(2n+1)xt}{\sin xt}dt$$

$$=\int e^{-t^2/x^2}.\frac{\sin(2n+1)t}{\sin t}\frac{dt}{x},\text{ by putting } t \text{ for } xt.$$

Let n become infinite, the value of the integral is

$\frac{\pi}{x}\left\{\tfrac{1}{2}+e^{-\pi^2/x^2}+e^{-4\pi^2/x^2}+\ldots\ldots\right\}$, *Bromwhich, Infinite Series.* Ex 4, p. 447,

$$\therefore\ \sqrt{\pi}\left\{\tfrac{1}{2}+\sum_{1}^{\infty}e^{-n^2x^2}\right\}=\frac{\pi}{x}\left\{\tfrac{1}{2}+\sum_{1}^{\infty}e^{n^2\pi^2/x^2}\right\};$$

writing $\pi\, x$ for x^2, we get

$$\tfrac{1}{2}+\sum_{1}^{\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}\left\{\ +\sum_{1}^{\infty}e^{-n^2\pi/x}\right\}.\quad \dots\qquad \dots\ (1)$$

Both sides are equal when $x=1$.

Differentiating with respect to x, we have,

$$\sum_{1}^{\infty}-e^{-n^2\pi x}.\,n^2\pi=-\frac{1}{2x^{\frac{3}{2}}}\left\{\tfrac{1}{2}+\sum_{1}^{\infty}e^{-n^2\pi/x}\right\}$$

$$+\frac{1}{\sqrt{x}}\left\{\sum_{1}^{\infty}e^{-n^2\pi/x}.\,\frac{n^2\pi}{x^2}\right\}.$$

Putting $x=1$

$$\sum_{1}^{\infty}e^{-n^2\pi}(2n^2\pi-\tfrac{1}{2})=\tfrac{1}{4},$$

$$\sum_{1}^{\infty}e^{-n^2\pi}\ (n^2\pi-\tfrac{1}{4})=\tfrac{1}{8}.$$

In (1) write $r\,(\cos\theta+i\sin\theta)$ for x and we see that

$$\tfrac{1}{2}+\sum_{1}^{\infty}e^{-n^2\pi r\cos\theta}.\cos\,(n^2\pi r\sin\theta)$$

$$=\frac{\cos\frac{\theta}{2}}{2\sqrt{r}}+\frac{1}{\sqrt{r}}R\left\{e^{-i\frac{\theta}{2}}\sum_{1}^{\infty}e^{-n^2\frac{\pi}{r}(\cos\theta-i\sin\theta)}\right\}$$

where $R\,(x)$ denotes the real part of x;

$$=\frac{\cos\frac{\theta}{2}}{2\sqrt{r}}+\frac{1}{\sqrt{r}}R\sum_{1}^{\infty}e^{-n^2\frac{\pi}{r}\cos\theta+i\left(n^2\frac{\pi}{r}\sin\theta-\frac{\theta}{2}\right)}$$

$$=\frac{\cos\frac{\theta}{2}}{2\sqrt{r}}+\frac{1}{\sqrt{r}}\sum_{1}^{\infty}e^{-n^2\frac{\pi}{r}\cos\theta}.\cos\left(n^2\frac{\pi}{r}\sin\theta-\frac{\theta}{2}\right)\quad\dots\qquad (2)$$

Put $r=1$, we obtain

$$\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n\pi\cos\theta}\cos(n^2\pi\sin\theta)$$

$$=\cos\frac{\theta}{2}\left\{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi\cos\theta}.\cos(n^2\pi\sin\theta)\right\}.$$

$$+\sin\frac{\theta}{2}\sum_{1}^{\infty} e^{-n^2\pi\cos\theta}\sin(n^2\pi\sin\theta).$$

Writing x for $\cos\theta$, $\sqrt{1-x^2}$ for $\sin\theta$, and simplifying we get

$$\frac{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi x}\cos(n^2\pi x\sqrt{1-x^2})}{\sum_{1}^{\infty} e^{-n\pi^2 x}\sin(n^2\pi\sqrt{1-x^2})}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}.$$

(a) Let $\dfrac{\cos\theta}{r}=1,\ \dfrac{\sin\theta}{r}=2$; then

$$r=\frac{1}{\sqrt{5}},\ r\ \cos\theta=\frac{1}{5},\ r\sin\theta=\frac{2}{5};$$

$$\tan\theta=2,\ \cos\frac{\theta}{2}=\frac{1}{5^{\frac{1}{4}}}\sqrt{\frac{\sqrt{5}+1}{2}};$$

making these substitutions in (2), we have

$$\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\frac{\pi}{5}}\cos\left(2n^2\frac{\pi}{5}\right)=5^{\frac{1}{4}}\ \cos\frac{\theta}{2}\left\{\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\pi}\right\};$$

when n is not a multiple of 5, $\cos\left(2n^2\frac{\pi}{5}\right)=\cos\frac{n\pi}{5}$; when n is a multiple of 5, $\cos\left(2n^2\frac{\pi}{5}\right)=1$.

$$\therefore\ \tfrac{1}{2}+\sum_{1}^{\infty} e^{-\frac{n^2\pi}{5}}\cos\left(2n^2\frac{\pi}{5}\right)$$

$$=\tfrac{1}{2}+\sum_{1}^{\infty} e^{-n^2\frac{\pi}{5}}\cos\frac{2\pi}{5}+\sum_{1}^{\infty} e^{5n^2\pi}\left(1-\cos\frac{2\pi}{5}\right).$$

By (1) $\frac{1}{2}+\sum_1^\infty e^{n^2\pi/5}=\sqrt{5}\left\{\frac{1}{2}+\sum_1^\infty e-\frac{n^2\pi}{5}\right\}$

$$\therefore \frac{1}{2}+\sum_1^\infty e^{-\frac{n^2\pi}{5}}\cos\left(2n^2\frac{\pi}{5}\right)$$

$$=\frac{1}{2}\left(1-\cos\frac{2\pi}{5}\right)+\sqrt{5}\cos\frac{2\pi}{5}\left\{\frac{1}{2}+\sum_1^\infty e^{-5n^2\pi}\right\}$$

$$+\sum_1^\infty e^{-5n^2\pi}\left(1-\cos\frac{2\pi}{5}\right)$$

$$=\left(1-\cos\frac{2\pi}{5}+\sqrt{5}.\cos\frac{2\pi}{5}\right)\left\{\frac{1}{2}+\sum_1^\infty e^{-5n^2\pi}\right\}$$

$$\therefore \frac{\frac{1}{2}+\sum_1^\infty e^{-\pi n^2}}{\frac{1}{2}+\sum_1^\infty e^{-5\pi n^2}}=\frac{1+\cos\frac{2\pi}{5}\left(\sqrt{5}-1\right)}{5^{\frac{1}{4}}\cos\frac{\theta}{2}}=\frac{1+\frac{(\sqrt{5}-1)^2}{4}}{\sqrt{\frac{\sqrt{5}+1}{2}}}$$

$$=\frac{(10-2\sqrt{5})}{4}\sqrt{\frac{\sqrt{5}-1}{2}}=\sqrt{5}.\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3}{2}}=\sqrt{5}\sqrt{\sqrt{5}-2}$$
$$=\sqrt{5\sqrt{5}-10}.$$

(b) $\frac{1}{2}+\sum_1^\infty e^{-\pi n^2 x}\cos\left(\pi n^2\sqrt{1-x^2}\right)$

$$=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}\sum_1^\infty e^{-\pi n^2 x}\sin\left(\pi n^2\sqrt{1-x^2}\right).$$

Since $\underset{x=1}{\mathrm{Lt}}\frac{\sin\left(\pi n^2\sqrt{1-x^2}\right)}{\sqrt{1-x}}=\underset{x=1}{\mathrm{Lt}}\sqrt{1+x}\,\frac{\sin\left(\pi n^2\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}=\sqrt{2}.\pi n^2;$

we get, by putting $x=1$,

$$\frac{1}{2}+\sum_1^\infty e^{-\pi n^2}=4\pi\sum_1^\infty e^{-\pi n^2}.n^2$$

Hence $$\sum_1^\infty e^{-\pi n^2}\left(\pi n^2-\frac{1}{4}\right)=\frac{1}{8},$$

a result already obtained directly by differentiating (1).

Question 665.

(R. N. Apte, M. A., F. R. A. S.) :—Find the value of

$$\iint xyz\left\{1+\overline{\frac{\partial z}{\partial x}}\Big|^2+\overline{\frac{\partial z}{\partial y}}\Big|^2\right\}^{\frac{1}{2}}dxdy \text{ where } z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}} \text{ and}$$

the integration is over the positive quadrant of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution by K. B. Madhava and H. V. Venkataramiengar.

Since $\frac{dz}{dx}=-\frac{c^2x}{a^2z}+\frac{dz}{dy}=-\frac{c^2y}{b^2z}$ the given integral

$$=\iint cxy\sqrt{1+\frac{c^2-a^2}{a^4}x^2+\frac{c^2-b^2}{b^4}y^2}\,dxdy,$$

where the integration extends to all positive values of $x+y$ subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$. Put $\xi=x^2/a^2$, $\eta=y^2/b^2$;

$$I=c.\frac{a^2b^2}{4}\iint\left(1+\frac{c^2-a^2}{a^2}\xi+\frac{c^2-b^2}{b^2}\eta\right)^{\frac{1}{2}}d\xi d\eta \text{ where } \xi+\eta<1$$

$$=\frac{2}{3}\cdot\frac{c.a^2b^2}{4}\cdot\frac{a^2}{c^2-a^2}\int_0^1\left\{\frac{c^2}{a^2}-c^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\eta\right\}^{\frac{3}{2}}-\left\{1+\frac{c^2-b^2}{b^2}\eta\right\}^{\frac{1}{2}}d\eta$$

$$=\frac{c}{15}\cdot\frac{1}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}\left[a^2(c^5-b^5)(a^2-b^2)-c^2(c^2-b^2)(a^5-b^5)\right].$$

Question 666.

(S. Ramanujan) :—Solve in positive rational numbers $x^y=y^x$.
For example : $x=4$, $y=2$; $x=3\frac{3}{8}$, $y=2\frac{1}{4}$.

Solution by J. C. Swaminarayan, M.A., and R. Vythynathaswamy.

Put $x=ky$. Then $y^k=ky$.

$$\therefore \qquad y^{k-1}=k.$$

The solution will be a rational number only if k is of the form $1+\frac{1}{n}$.

$$\therefore \qquad y=\left(1+\frac{1}{n}\right)^n \text{ and } x=\left(1+\frac{1}{n}\right)^{n+1}.$$

When $n=1$, $x=4$, $y=2$; when $n=2$, $x=\frac{27}{8}$, $y=\frac{9}{4}$ when $n=3$,

$$x=\frac{256}{81}, y=\frac{64}{27}.$$

Thus infinite solutions in positive rational integers will be obtained corresponding to all the integral values of n.

Question 667.

(S. Narayana Aiyar, M.A.) :—Establish the truth of the following theorems:

1. If $F(y)=\phi(y)-\frac{x}{1!}\phi(y+1)+\frac{x^2}{2!}\phi(y+2)-\frac{x^3}{3!}\phi(y+3)+\ldots$

then $\phi(y)=F(y)+\frac{x}{1!}F(y+1)+\frac{x^2}{2!}F(y+2)+\frac{x^3}{3!}F(y+3)+\ldots$

2. If $F(y)=\phi(y)-\frac{n}{1}\phi(y+1)+\frac{n(n+1)}{1\cdot 2}\phi(y+2)-\ldots$

then $\phi(y)=F(y)+\frac{n}{1}F(y+1)+\frac{n(n-1)}{1\cdot 2}F(y+2)+\ldots$

Solution (1) *by J. C. Swaminarayan, M.A., and R. Srinivasan M.A.*; (2) *by K.B. Madhava.*

(1) Let E be an operator which when applied to a function of y changes y into $y+1$.

Then $F(y)=\left\{1-\frac{xE}{1!}+\frac{x^2E^2}{2!}-\frac{x^3E^3}{3!}+-\right\}\phi(y)$

$$=e^{-xE}\phi(y)$$

$$\therefore \qquad \phi(y)=e^{xE}F(y)$$

$$=\left\{1+\frac{xE}{1!}+\frac{x^2E^2}{2!}+\frac{x^3E^3}{3!}+\ldots\ldots\right\}F(y)$$

$$=F(y)+\frac{x}{1!}F(y+1)+\frac{x^2}{2!}F(y+2)+\frac{x^3}{3!}F(y+3)+\ldots$$

Also $F(y)=\left\{1-\frac{nE}{1}+\frac{n(n+1)E^2}{1\cdot 2}-\ldots\ldots\right\}\phi(y)$

$$=(1+E)^{-n}\phi(y).$$

$$\therefore \qquad \phi(y)=(1+E)^nF(y)$$

$$=\left\{1+\frac{n}{1}E+\frac{n(n+1)}{1\cdot 2}E^2+\ldots\right\}F(y)$$

$$=F(y)+\frac{n}{1}F(y+1)+\frac{n(n-1)}{1\cdot 2}F(y+2)+\ldots\ldots$$

(2) Substituting in the right hand side expressions for $F(y)$ $F(y+1)$ etc., in terms of $\phi(y)$, $\phi(y+1)$...in the first part, we have for the co-efficient of $\phi(y+r)$

$$\frac{x^r}{r!}\left\{1-{}_rC_1+{}_rC_2-\ldots 1\right\}=0$$

except in the case of the first term, which is unity, whence the identity.

Substituting similarly, in the second part the co-efficient of $\phi\,(y+r)$

$$\frac{n(n+1)\ldots(n+r-1)}{r\,!}-\frac{n}{1}\cdot\frac{n(n+1)\ldots(n+r-2)}{(r-1)\,!}$$
$$+\frac{n(n-1)}{1\cdot2}\frac{n(n+1)\ldots(n+r-3)}{(r-2)\,!}-\ldots$$

which is the co-efficient of x^n in the product of $(1-x)^{-n}$ and $(1-x)^n$ and is zero.

The co-efficient of $\phi\,(y)$ however is unity, ; hence the result.

Question 677.

(D. Krishnamurti) :—A chain of lenth l and mass M is coiled at the edge of a table. A particle of mass m is fastened to one end of the chain and the other end is gently let slip over the edge of the table. Show that the velocity of the particle immediately after it leaves the table is $k\sqrt{[2\,gl\,(1+k^2)/3]}$, where $k=\mathrm{M}/(m+\mathrm{M})$.

Solution by K. B. Madhava and Martyn M. Thomas.

A particular case of this where $\mathrm{M}=m$ is given in Loney, p. 133 Ex. 11. [See also Routh, § 150.]

From Newton's Second Law as applied to motion where the moving mass varies, we have

$$\frac{d}{dt}\left(\frac{\mathrm{M}}{l}x\dot{x}\right)=g.\frac{\mathrm{M}}{l}.x. \qquad \ldots \qquad \ldots \quad (1)$$

To integrate this multiply by $\frac{\mathrm{M}}{l}x\dot{x}$, and then we have

$$\frac{1}{2}\cdot\left(\frac{\mathrm{M}}{l}x\dot{x}\right)^2=g.\frac{\mathrm{M}^2}{l^2}\cdot\frac{x^3}{3} \qquad \ldots \qquad \ldots \quad (2)$$

$$\therefore \qquad \dot{x}^2=\frac{2g}{3}l. \qquad \ldots \qquad \ldots \quad (3)$$

Calling this velocity u, we have by the principle of conservation of momentum, for the horizontal and vertical velocities

$$(\mathrm{M}+m)v=\mathrm{M}u \text{ and } (\mathrm{M}+m)w=\mathrm{M}v \qquad \ldots \qquad \ldots \quad (4)$$

i.e. $$v=ku \text{ and } w=k^2u \qquad \ldots \qquad \ldots \qquad \ldots \quad (5)$$

The sq. of the vel. when the particle has just left the table is

$$v^2+w^2=k^2(1+k^2)u^2=\tfrac{2}{3}gl(1+k^2) \text{ from (2)}.$$

Question 683.

(GANPATRAM R. JANI) :—If S_r is put for $1^r+2^r+3^r+\ldots n^r$, prove that

$$\begin{vmatrix} S_o & S_1 & \ldots & S_{n-1} \\ S_1 & S_2 & & S_n \\ \ldots & \ldots & & \\ S_{n-1} & S_n & & S_{2n-2} \end{vmatrix} = \left\{ \frac{(n\,!)^n}{1^1 2^2 3^3 \ldots n^n} \right\}^2$$

Solution by K. B. Madhava, A. Narasinga Rao, K. R. Rama Aiyar and Martyn M. Thomas.

This is just a particular case of the example due to Sylvester solved on p. 229 of Boole's *Finite Differences*, (1880) and the given determinant is, as shown there, seen to be equivalent to

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 3 & & n \\ 1^2 & 2^2 & 3^2 & & n^2 \\ 1^n & 2^n & 3^n & & n^n \end{vmatrix}^2$$

and this is equal to the product of the squares of the differences $1, 2, \ldots n$ taken with the proper sign.

Hence
$$D = -\left\{ \frac{(n\,!)^n}{1^1 2^2 3^3 \ldots n^n} \right\}^2$$

It is also shown in the reference (*loc. cit.*) that

$$\begin{vmatrix} S_x & S_{x+1} & \ldots & S_{x+n-1} \\ S_{x+1} & S_{x+2} & \ldots & S_{x+n-2} \\ S_{y+n-1} & S_{x+n} & \ldots & S_{x+n} \end{vmatrix} = (n\,!)^x D$$

Question 686.

(S. NARAYANA AIYAR, M.A.):—Establish the following identities :—

(i) $$\frac{a(a+1)(a+2)\ldots(a+n-1)}{b(b+1)(b+2)\ldots(b+n-1)} = 1 + {}_nC_1 . \frac{a-b}{b}$$
$$+ {}_nC_2 . \frac{(a-b)(a-b-1)}{b(b+1)} + \ldots + \frac{(a-b)(a-b-1)\ldots(a-b-n+1)}{b(b+1) \quad \ldots \quad (b+n-1)};$$

(ii) $$\frac{(a-b)(a-b-1)\ldots(a-b-n+1)}{b(b+1) \quad \ldots \quad (b+n+1)} = (-1)^n \left\{ 1 - {}_nC_1 \frac{a}{b} \right.$$
$$\left. + {}_nC_2 \frac{a(a+1)}{b(b+1)} - \ldots (-1)^n \frac{a(a+1)\ldots(a+n-1)}{b(b+1)\ldots(b+n-1)} \right\}$$

(iii) $F(a,b,c,x)=(1-x)^{-a}\,F\left(a,\,c-b,\,c,\frac{x}{x-1}\right)$, where F stands for the hypergeometric series,

(iv) (v) and (vi).

Remarks by K. B. Madhava and R. Srinivasan, M.A.

(i) is proved in the solution of Q. (616) ;

(ii) change a into $b-a$ in (i) and the result follows ;

(iii) is proved in the solution to Q. (616) ;

(iv) if in Q. (616) ϕ (Q)=log $(1+x)$, the result follows ;

(v) if in Q. (616) $\phi(x)=\sin x$ and $x=\pi$, the result follows ;

(vi) if in Q. (616) $\phi(n)=$ los x and $x-\pi$, the result follows ;

In connection with (iii) the following algebraic proof extracted from the *Messenger of Mathematics*, Vol. XLIV, No. 8, December 1914, may be of interest. The proof is by Prof. M. J. M. Hill, University College, London :

The righthand side is $(1-x)^{-a}\,F\left(a,c-b,c,\frac{x}{x-1}\right)$

$$=(1-x)^{-a}+\frac{a(c-b)}{1.c}(-x)(1-x)^{-a-1}$$

$$+\frac{a_2.(c-b)_2}{2.c_2}(-x)^2(1-x)^{-a-2}+\ldots$$

$$+\frac{a_n(c-b)_n}{n\,!\,c_n}(-x)^n(2-x)^{-a-n}.$$

Hence the coeff. of x^n is

$$\frac{a_n}{n!}+\frac{a.c-b}{1.c}.(-)\frac{(a+1)_{n-1}}{(n-1)\,!}+\frac{a_2.(c-b)_2}{2\,!\,c_2}.(-)^2.\quad\frac{(a+2)_{n-2}}{n-2\,!}$$

$$\ldots(-)^n\frac{a_n(c-b)_n}{n\,!\,c_n}$$

$$=\frac{a_n}{n\,!\,c_n}\{\,c_n-nc_1.\,(c-b)(c+1)_{n-1}+nc_2(c-b)_2\,(c+2)_{n-2}$$

$$+\ldots(-)^n(c-b)_n\,\}$$

The $(r+1)^{th}$ term in brackets is

$(-)^r nc_r(c-b)_r(c+r)_{n-r}$

$$=(-1)^r nc_r(c-b)(c-b-1)\ldots(c-b+r-1)\times$$

$$(c+r)(c+r+1)\ldots(c+n-1)$$

$$=nc_r(b-c)(b-c-1)\ldots(b-c-r+1)\times(c+n-1)(c+n-2)\ldots(c+r).$$

If we now adopt, as in the statement of Vandermonde's theorem, an abbreviation for $a\,(a-1)\ldots(a-r+1)$, say $\overline{a}_r$, so that V's theorem can be written

$$\overline{(a+b)}_n = \sum_{r=0}^{n} {}^nc_r \overline{a}_r \overline{b}_{n-r}$$

we have for the $(r+1)^{th}$ term in brackets the expression

$${}^nc_r \overline{(b-c)}_r \overline{(c+n-1)}_{n-r}$$

and therefore by Vandermonde's theorem, the whole expression in brackets

$$=\overline{(b-c+c+n-1)}_n$$

$$=\overline{(b+n-1)}_n$$

$$=(b+n-1)(b+n-2)\ldots(b+1)b.$$

$=b_n$, in the notation of this paper, where throughout we have adopted for abbreviation c_n for $c\,(c+1)(c+2)\ldots(c+n-1)$,

and therefore the coeft. of x^n is $\dfrac{a_n b_n}{n\,!\,c_n}$ which is the coeft. of x^n in $F(a,b,c,x)$.

Thus the identity is demonstrated.

Question 696.

(S. Krishnasawmi Aiyangar) :—If λ, μ be the latera-recta of the parabola and rectangular hyperbola of closest contact with a curve at any point, prove that $2\lambda\rho=\mu^2$.

Solution by R. Srinivasan, M.A., Martyn M. Thomas and K. B. Madhava.

With the usual notation, in a parabola

$$\frac{SP}{SY}=\frac{SY}{SA}=\sec\phi.$$

$$\therefore \qquad SP=SA\sec^2\phi.$$

$$\therefore \qquad \rho=2a\sec^3\phi.\quad (SA=a).$$

Also in a rectangular hyperbola,

$$\rho=\frac{CP^3}{CY},\ \frac{CP}{CY}=\sec\phi,\ CD.\,CY=\alpha^2,$$

where α is the semi-axis (or semi latus-rectum.)

$$\therefore \qquad \rho=\alpha\sec^{\frac{3}{2}}\phi.$$

$$\therefore \qquad \lambda=2\rho\cos^3\phi \text{ and } \mu=2\rho\cos^{\frac{3}{2}}\phi.$$

$$\therefore \qquad 2\lambda\rho=\mu^2.$$

Question 703.

(N. SANKARA AIYAR, M.A.):—If AP is the symmedian through A, prove that

$$\Sigma\{(b^2+c^2)(AK.\ KP)\}=3a^2b^2c^2/(a^2+b^2+c^2).$$

Solution by R. Srinivasan, M.A., K. B. Madhava, R. D. Karve and L. N. Subramanian.

Let the distance of K from BC be x, and the attitude AD be p.

Now $$BP : PC :: c^2 : b^2.$$

$\therefore$ $$b^2\,AB^2+c^2.AC^2=b^2.BP^2+c^2CP^2+(b^2+c^2)AP^2.$$

i.e. $$AP^2=\frac{b^2c^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2}. \quad \ldots \quad \ldots \text{(i)}$$

Again $$\frac{KP}{AP}=\frac{x}{p}=\frac{2a\Delta/(a^2+b^2+c^2)}{2\Delta/a}$$

$$=\frac{a^2}{a^2+b^2+c^2}.$$

and $$\frac{AK}{AP}=\frac{b^2+c^2}{a^2+b^2+c^2}.$$

$\therefore$ $$AK\cdot KP=\frac{a^2(b^2+c^2)}{(a^2+b^2+c^2)^2}AP^2.$$

$\therefore$ $$(b^2+c^2)\,AK.KP=\frac{a^2b^2c^2}{(a^2+b^2+c^2)^2}(2b^2+2c^2-a^2).$$

$\therefore$ $$\Sigma\{(b^2+c^2)\,AK\cdot KP\}=\frac{3\,a^2b^2c^2}{a^2+b^2+c^2}.$$

Question 704.

(S. MALHARI RAO, B.A.):—If the sum of a number of 3 digits and the number formed by reversing the digits be divisible by 37, the sum of all such pairs of numbers is 480×37.

Solution by S. V. Venkatarayasastri, M.A., L.T. and R. D. Karve.

If a, b, c be the 3 digits, the sum of the number and the number formed by reversing the digits$=101\,(a+c)+20b$.

If this is divisible by 37, the values of b and $a+c$ are the 9 pairs 1, 2; 2, 4; 3, 6;......9, 18.

Taking the first set of values 1, 2 for b, $a+c$, we get 210 and 111. As these do not give us a pair of 2 numbers satisfying the given conditions, we reject them.

Taking the set 2, 4, we get 2 numbers, omitting numbers 420 and 222 for the above reason. Their sum$=222\times2$.

Taking the set 3, 6, we get 4 numbers or 2 pairs. Their sum$=333\times4$.

And so on.

The sum of all such pairs$=222\times2+333\times4+444\times4+\ldots+888\times2$

$$=480\times37.$$

QUESTIONS FOR SOLUTION.

725. (K. B. Madhava) :—Show that

$$\int^{\infty} \frac{x^3 dx}{1+x^9 \sin^2 x} \text{ converges,}$$

but that

$$\int^{\infty} \frac{x^3 dx}{1+x^8 \sin^2 x} \text{ diverges.}$$

726. (K. B. Madhava) :—Shew that

$$\sum_{n=1}^{\infty} \left\{ \frac{\Gamma(n)}{\Gamma(n+4)} \right\}^2 = \frac{1}{6^3} (20\pi^2-197) ;$$

and that

$$\sum_{1}^{\infty} \left\{ \frac{\Gamma(n)}{\Gamma(n+5)} \right\}^2 = \frac{1}{12^4 . 2^2} (1680\pi^2-16575).$$

727. (Communicated by Mr. P. V. Seshu Aiyar) :—Three persons A, B, C had a, b, c mangoes respectively. A gave B and C $\frac{b(b+1)}{2}$, and $\frac{c(c+1)}{2}$ mangoes respectively out of what he had; then B gave C and A similarly; and finally C gave A and B similarly. It was found that A, B, and C had the same number of mangoes ultimately. How many had each at first ?

728. (K. Appukuttan Erady, M.A.) :—If $u \equiv (abcfgh)(xyz)^2$, show that

$$\iiint u^n \, dx \, dy \, dz$$

taken throughout the space bounded by $u=1$, is

$$\frac{4\pi}{2n+3} \Delta^{-\frac{1}{2}}$$

where Δ is the discriminant of u.

729. (K. Padmanabhulu, B.A.) :—If the earth were to break up into an indefinite number of fragments at any point in its course round the sun by any sudden explosion, prove that all the fragments meet again at the same point; and that at the middle of the interval between the explosion and junction all the pieces will be moving with equal velocities in parallel directions.

730. (S. Krishnaswami Aiyangar) :—Shew that the locus of the orthopoles of tangents to the maximum inscribed ellipse of a triangle is a straight line through the ortho-centre.

731. (S. KRISHNASWAMI AIYANGAR) :—In a spherical triangle prove that

$$\begin{vmatrix} \sin a & \sin b & \sin c \\ \operatorname{cosec} a & \operatorname{cosec} b & \operatorname{cosec} c \\ \operatorname{cosec}^3 a \operatorname{cosec} A & \operatorname{cosec}^3 b \cos B & \operatorname{cosec}^3 c \operatorname{cosec} C \end{vmatrix}$$

is equal to

$$\operatorname{cosec}^3 a \operatorname{cosec}^3 b \operatorname{cosec}^3 c (\cos a-\cos b)(\cos b-\cos c)(\cos c-\cos a)\times (1-\cos a-\cos b-\cos c).$$

What is the corresponding formula in a plane.

732. (R. VYTHYNATHASWAMY) :—There are two kinds of elements α, β, so related that n elements of either kind determine uniquely an element of the other kind. Prove that the aggregate of each must be n-dimensional. Further, supposing

β_r to be the α-element determined by $(\alpha_{r1}, \alpha_{r2}, \ldots \alpha_{rn})$ $(r=1,2\ldots n+1)$

$\beta_{p,r}$ the element determined by $(\alpha_{p1}, \alpha_{p2} \ldots \alpha_{pr-1}, \alpha_{pr+1} \ldots; \alpha_{pn+1})$ $(p=1,2\ldots n)$

α_r the element determined by $(\beta_{1,r}, \beta_{2,r}, \ldots \beta_{n,r})$ $(r=1,2\ldots n+1)$;

shew that if the $(n+1)$ elements β_r determine the same α-element, then the $(n+1)$ elements α_r determine the same β-element and conversely.

733. (R. VYTHYNATHASWAMY) :—Required a definition of continuity which does not involve the idea of number; also a definition or explanation and not pre-supposing parameters of 'dimension' as applied to a dense aggregate of similar elements.

Does the idea of dimension involve the idea of continuity or even that of being dense?

Is the aggregate of all numbers, real and complex, to be regarded as one-dimensional or two-dimensional?

734. (J. C. SWAMINARAYAN, M.A.) :—Solve the differential equation

$$\left(y^2-b^2\right)\frac{dy}{dx}+xy=\sqrt{b^2x^2+a^2y^2-a^2b^2}.$$

735. (SELECTED) :—A, B, C, D, E represent the entire circumferences of a curve and its successive pedals. If C pertains to an ellipse having its centre at the pedal origin, show that

$$B(B+D)=(2C-E)(3C-A). \qquad \text{(M.A. 1903, Madras.)}$$

736. (R. SRINIVASAN, M.A.) :—Shew that the common tangent to the nine-point and inscribed circles of a triangle ABC cuts the sides a, b, c in the ratios

$$\frac{a-b}{a-c}, \frac{b-c}{b-a}, \frac{c-a}{c-b}.$$

737. (E. H. NEVILLE):—A continuous function of position f which is nowhere negative is defined for every point of a closed plane curve C which does not extend to infinity; m is a positive number, and B_m is the region of the plane containing every point whose distance from some point of C is less than m times the value of f for that point of C (in other words, a circle is described round each point of C with radius mf, and a point belongs to B_m if it is in the interior of one of the circles); shew that if C has everywhere a definite tangent and a finite curvature and if m is sufficiently small, the area A_m of the region B_m is

$$2m\int\{\cos\alpha-\tfrac{1}{2}m^2f(d^2f/ds^2)\sec\alpha\}fds,$$

where α is the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is $m\,df/ds$, and the integral is taken along C. Find a corresponding expression valid if C has end-points, and further shew that if C is composed of any finite number of distinct arcs and has cusps, multiple points, and vertices, then provided that these singular points and the end-points are finite in number the quotient $A_m/2m$ tends to $\int fds$ as m tends to zero.

738. (S. RAMANUJAN):—If $\phi(x)=e^{-x}+\frac{x}{1!}e^{-2x}+\frac{3x^2}{2!}e^{-3x}+$

$$\frac{4^2x^3}{3!}e^{-4x}+\frac{5^3x^4}{4!}e^{-5x}+\ldots,$$

show that $\phi(x)=1$ when x lies between 0 and 1; and $\phi(x)\neq 1$ when $x>1$. Find the limit of

$$\frac{\phi(1+E)-\phi(1)}{E}$$

as $E\to 0$ by positive values.

739. (S. RAMANUJAN):—Show that

$$\int_0^\infty e^{-nx}(\cot x+\coth x)\sin nx\,dx=\frac{\pi}{2}\left(\frac{1+e^{-\pi^n}}{1-e^{-\pi^n}}\right)^{(-1)^n}$$

for all positive integral values of n.

740. (S. RAMANUJAN):—If

$$\phi(x)=\left\{\frac{e^x[x]^2}{x^{[x]}}\right\}-2\pi x,$$

where $[x]$ denotes the greatest integer in x, show that $\phi(x)$ is a continuous function of x for all positive values of x and oscillates from $\frac{\pi}{3}$ to $-\frac{\pi}{6}$ when x becomes infinite. Also differentiate $\phi(x)$.

Statement of Accounts of the Indian Mathematical Society for the year 1915.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To opening balance	508	10	2	By books and Journals	332	10	2
„ subscriptions from members	2,294	7	0	„ Library	293	0	0
„ „ for Journal	245	4	0	„ printing Journal	477	3	6
„ miscellaneous receipts	41	1	0	„ ordinary working expenses	202	12	8
				„ closing balance*	1,783	11	10
	3,089	6	2		3,089	6	2

* Fixed deposit with the

	Rs.	A.	P.
Madras Central Urban Bank	800	0	0
Treasurer	839	11	7
Secretary	52	9	0
Asst. Secretary	55	3	3
Asst. Librarian	36	4	0
	1,783	11	10

I have examined the Treasurer's books and vouchers and the monthly statements submitted by the Secretary, Asst. Secretary and Asst. Librarian and declare the above accounts to be correct.

C. N. Ganapathi Aiyar.

Madras,
5th February 1916.

S. Narayana Aiyar,
Hony. Treasurer.

List of Periodicals Received.

(From 16th November 1915 to 15th January 1916)

1. Acta Mathematica, Vol. 40, Nos. 1 & 2.
2. Astrophysical Journal, Vol. 42, No. 3, October 1915.
3. Bulletin of the American Mathematical Society, Vol. 22, No. 2, November 1915.
4. Bulletin des Sciences Mathematiques, Vol. 39, October 1915.
5. Educational Times, December 1915, (6 Copies).
6. L'Intermediaire des Mathematiciens, Vol. 22, No. 10, October 1915.
7. Mathematical Gazette, Vol. 8, No. 119, October 1915, (4 Copies).
8. Mathematics Teacher, Vol. 8, No. 1, September 1915.
9. Messenger of Mathematics Vol. 45, Nos. 4 and 5, August and September 1915.
10. Philosophical Magazine, Vol. 30, Nos. 179 and 180, November and December 1915.
11. Popular Astronomy, Vol. 23, Nos. 9 and 10, November and December 1915, (3 Copies)
12. Proceedings of the Edinburgh Mathematical Society Vol. 33, Parts 1 and 2, Session 1914—1915.
13. Proceedings of the London Mathematical Society, Vol. 14, Nos. 6 and 7, October and November 1915.
14. Proceedings of the Royal Society of London, Vol. 92, Nos. 634, 635 and 636.
15. Quarterly Journal of Mathematics, Vol. 46, No. 4, October 1915.
16. School Science and Mathematics, Vol. 15, Nos. 8 and 9, November and December 1915, (2 Copies).
17. Transactions of the American Mathematical Society, Vol. 16, No. 4, October 1915.
18. Transactions of the Royal Society of London, Vol. 216, Nos. 538 and 539.
19. Tohoku Mathematical Journal, Vol. 8, No. 2, October 1915.
20. Rendiconti Del Circolo Mathematics Di Palermo, Vol. 39, No. 3.
21. American Mathematical Monthly, Vol. 22, No. 8, October 1915.

The Indian Mathematical Society

(Founded in 1907 for the Advancement of Mathematical Study and Research in India.)

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THE JOURNAL

OF THE

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Vol. VIII.] APRIL 1916. [No. 2.

EDITED BY

M. T. NARANIENGAR, M. A.

Hony. Joint Secretary,

WITH THE CO-OPERATION OF

Prof. R. P. PARANJPYE, M.A., B.Sc. Prof. A. C. L. WILKINSON, M.A., F.R.A.S.

and others.

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CONTENTS.

The Journal is open to contributions from members as well as subscribers. The editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. R. KRISHNASWAMI, M.A., 3, Johannes Road, Saidapet, Madras.

Enquiries by intending members and all other communications may be addressed to the Hony. Joint Secretary, D. D. KAPADIA, M.A., B.Sc., 2414, East Street, Poona.

THE JOURNAL
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Vol. VIII.] APRIL 1916. [No. 2.

PROGRESS REPORT.

1. The following books have been purchased for the Library :—

1. *The Development of Mathematics in China and Japan* by Y. Mikami, B. G. Teubner, Dresden, Leipzig 1913. Price Rs. 16-10.

2. *Euclid's Books on Divisions of Figures* by Prof. R.C. Archibald, Ph. D. Cambridge University Press 1915. Price Rs. 5-4.

2. The following gentleman has been elected a *Life Member* of our Society—

Mr. C. P. Ramaswami Aiyar, B.A., B.L.—Vakil, High Court, and Fellow, Madras University, Madras.

3. The following gentlemen have been elected members of the Society.

1. *Mr. S. Ramanujan*—at present studying at Cambridge, (at concessional rate):
2. *Mr. F. H. V. Gulasegaram, B.A.*—Chief Lecturer in Mathematics, St. John's College, "Sangli Tope," Nallur, Jaffna (Ceylon).
3. *Mr. J. M. Bose, M.A., B.Sc., (Edin), Bar at Law*—Professor of Mathematics, Govt. Coll., Chittagong, Bengal.

4. The audited balance sheet of the Accounts for the year 1915, was published with the last issue of our Journal. The Budget for the current year is published herewith for the information of the General Body. It is a matter for congratulation that both the last year's balance and the present budget are satisfactory.

POONA,
31st March 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

Budget for 1916.

Receipts—		Rs.	A.	P.	*Charges—*		Rs.	A.	P.
Opening Balance	...	1,783	11	10	Books and Journals	...	400	0	0
Subscriptions from members	...	1,350	0	0	Library charges	...	350	0	0
Sale of Journals	...	200	0	0	Printing Journal	...	600	0	0
Miscellaneous	...	50	0	0	Ordinary working expenses	...	250	0	0
					Closing Balance	...	1,783	11	10
Total	...	3,383	11	10	Total	...	3,383	11	10

MADRAS,
16th February 1916.

S. NARAYANA AIYAR, M.A., F.S.S.,
Hony. Treasurer.

Stability and Oscillations of Plane Kites.

By J. M. Bose, M.A., B.Sc.

1. In the Bulletin of the Calcutta Mathematical Society, Vol. II Pt. 1, I showed that the investigation of the stability and small oscillations of a plane kite, can be reduced to that of the roots of a certain cubic. But the results obtained in that paper were based on the assumption that the tension of the string acted at a point on the axis of symmetry, whose distance h from the origin remained constant during the small motions which were impressed. It has recently been pointed out to me by Prof. Bryan, that that this condition does not as a rule hold good, and if the variation of h be taken into account, the condition of stability would be somewhat different from those given in that paper.

In the present paper it is proposed to remove this restriction, and to give a more general investigation of the conditions of stability of plane kites.

Take the centre of gravity of the kite as origin, and as moving axes take the line joining the "head" and the "tail" (*i.e.*, the axis of symmetry, as y-axis, and a line lying in the plane of the kite as x axis, and finally a line perpendicular to them (*i.e.*, perpendicular to the plane of the kite) as z-axis. The axes are thus fixed relatively to the body of the kite and are moving with it.

Let A, B, C be the moments of inertia, $u, v, w, \theta_1, \theta_2, \theta_3$, the linear and angular velocities of the kite. The products of inertia evidently vanish since the kite is assumed to be a plane with an axis of symmetry.

Let us suppose that initially the plane of the kite is vertical; and let its plane be rotated round the axis of symmetry through an angle ψ, this brings the axis Oz to the position Oz_1; in this position let the plane of the kite be again rotated round Oz_1 through an angle θ, and finally let the plane of the kite be rotated round the x-axis (in its new position) through an angle ϕ.

The generalised angular co-ordinates θ, ϕ, ψ, are connected with the angular velocities $\theta_1, \theta_2, \theta_3$ by the following Euler's geometrical equations

$$\theta_1=\dot{\phi}+\dot{\psi}\sin\theta$$

$$\theta_2=\dot{\theta}\sin\phi+\dot{\psi}\cos\theta\cos\phi$$

$$\theta_3=\dot{\theta}\cos\phi-\dot{\psi}\cos\theta\sin\phi.$$

The impressed forces on the kite are

(i) the air resistance which (in the case of planes,) may be taken to be a single force R, acting at a point on the axis of symmetry called the centre of pressure. Let its co-ordinates be $(0, \eta, 0)$.

(ii) the weight W acting at the centre of gravity; its direction cosines with reference to the moving axes being $\sin\theta$, $\cos\theta\cos\phi$, $\cos\theta\sin\phi$.

(iii) the tension T of the string, acting in directions whose cosines are l, m, n at the instant under consideration. As a rule, the string bifurcates at a point, and the two parts are attached to two different points on the axis of symmetry, but we can always replace the two tensions by their resultant. Let the co-ordinates of the point of application of this resultant be (f, h, o).

We shall for the present suppose the string to be very long (almost infinite) so that during the small oscillations of the kite the direction of the string in space, as well as the magnitude of the tension, may both be assumed to be absolute constants.* The case of *finite* string will be dealt with afterwards.

If χ be the inclination of the string to the vertical, supposed constant, then it can be easily proved from considerations of elementary solid geometry, that the direction and point of application of the tension are given by

$$l=-(\cos\chi\sin\theta+\sin\chi\sin\psi\cos\theta)$$
$$m=-\cos\chi\cos\theta\cos\phi+\sin\chi(\cos\psi\sin\phi+\sin\psi\sin\theta\cos\phi)$$
$$n=-\cos\chi\cos\theta\sin\phi+\sin\chi(\cos\psi\cos\phi-\sin\psi\sin\theta\sin\phi)$$
$$f=h_o\frac{\sin\overline{(\phi+\chi)}}{\sin\chi}\left(\sin\theta+\frac{l}{n}\cos\theta\sin\phi\right)$$
$$h=h_o\frac{\sin\overline{\phi+\chi}}{\sin\chi}\left(\cos\theta\cos\phi+\frac{m}{n}\cos\theta\sin\phi\right)$$

h_o being the initial value of h, and the quantities l, m, n, f, h are referred to the axes in their new position.

Let X, Y, Z be the components of the total force, L, M, N their moments about the axes, and T the constant tension of the string, then we have

$$X=Tl+W\sin\theta$$
$$Y=Tm-W\cos\theta\cos\phi$$
$$Z=Tn+W\cos\theta\sin\phi-R$$
$$L=hTn-\eta R$$
$$M=-fTn$$
$$N=T(fm-hl).$$

* I am indebted to Prof. Bryan for this suggestion.

In the position of equilibrium let the plane of the kite make an angle ϕ_o with the vertical, *i.e.*, the plane of the kite is rotated through an angle ϕ_o round the x-axis. Since the position of the old x—axis is unchanged, we have initially $\theta=0$, $\psi=0$, $\phi=\phi_o$.

Now all the variables involved in the small oscillations are functions of θ, ϕ, ψ, only. Hence, it follows from the above, that initially θ, ψ, f, l, are zero, and in the subsequent motion they remain small quantities of the first order.

Also, for small values of θ and ψ, we have

$$l=-\theta \cos \chi-\psi \sin \chi$$
$$m=-\cos \overline{\phi+\chi}$$
$$n=\sin \overline{\phi+\chi}$$
$$f=h_o(\theta \cos \phi_o-\psi \sin \phi_o)$$
$$h=h_o \frac{\sin \overline{\phi_o+\chi}}{\sin \overline{\phi+\chi}}$$

The conditions of equilibrium are

$$X_o=Tl_o=0$$
$$Y_o=Tm_o-W \cos \phi_o=0$$
$$Z_o=Tn_o+W \sin \phi_o-R_o=0$$
$$L_o=h_o Tn_o-\eta_o R_o=0$$
$$N_o=f_o m_o-h_o l_o=0.$$

2. *Equations of Motion*:—With the axes taken as above the equations of motion of the kite are

$$M\left(\frac{du}{dt}+w\,\theta_2-v\,\theta_3\right)=X=Tl+W \sin \theta$$
$$M\left(\frac{dv}{dt}+u\,\theta_3-w\,\theta_1\right)=Y=Tm-W \cos \theta \cos \phi$$
$$M\left(\frac{dw}{dt}+v\,\theta_1-u\,\theta_2\right)=Z=Tn+W \cos \theta \sin \phi-R$$
$$A\dot{\theta}_1+(C-B)\theta_2\theta_3=L=hTn-\eta R$$
$$B\dot{\theta}_2+(B-A)\theta_1\theta_3=M=-fTn$$
$$C\dot{\theta}_3+(A-C)\theta_2\theta_1=N=T(fm-hl).$$

The quantities whose variations determine the small oscillations, are the velocity components and the co-ordinates f, h, η. We have already given their values in terms of ϕ, ψ and χ. Hence to find the variations of f, h, η to the first order, we put $\sin\theta=\theta$, $\sin\psi=\psi$, $\phi=\phi_o+\epsilon$ and also assume u, v, w, θ, &c., to be small, so that their squares and products may be neglected.

† Throughout this paper zero suffixes are used to denote the initial values of the quantities involved.

Hence substituting these values and simplifying, we have

$$M\dot{u} = -T(\theta\cos\chi+\psi\sin\chi)+W\theta \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$M\dot{v} = \underline{-T\cos\overline{\phi_0+\chi}-W\cos\phi_0}+\epsilon(T\sin\overline{\phi_0+\chi}+W\sin\phi_0) \quad \ldots \quad (2)$$

$$M\dot{w} = \underline{T\sin\overline{\phi_0+\chi}+W\sin\phi_0-R_0}+$$

$$\epsilon(\underline{T\cos\overline{\phi_0+\chi}+W\cos\phi_0})-\delta R \qquad \ldots \quad \ldots \quad (3)$$

$$A\dot{\theta}_1 = \underline{h_0\ T\sin\overline{\phi_0+\chi}-\eta_0 R_0}-R_0\delta\eta_c-\eta_0\delta R \qquad \ldots \quad \ldots \quad (4)$$

$$B\dot{\theta}_2 = -h_0\ T(\theta\cos\phi-\psi\sin\phi_0)\sin\overline{\phi_0+\chi} \qquad \ldots \quad \ldots \quad (5)$$

$$C\dot{\theta}_3 = h_0 T(\theta\sin\phi+\psi\cos\phi)\sin\overline{\phi_0+\chi} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The terms underlined disappear, since (as can be easily proved from elementary statics) they are conditions of equilibrium.

3. *Resistance derivatives.* We have next to find δR and $\delta\eta$. The resistance experienced by a lamina moving through air, is a function of the velocity of the air, and also of the inclination of its plane face (called the "angle of attack") to the horizon, in steady motion or equilibrium. Hence if V be the velocity of air and α the angle of attack, we may write

$$R = R\,(V\cos\alpha,\ V\sin\alpha) = R\ (v_0,\ w_0)\ \text{say,}$$

the functional notation R on the right hand side being at present arbitrary. When the velocity components $u, v, \ldots \theta_3$ are impressed, we have

$$R_0+\delta R = R(u,\ v_0+v,\ w_0+w,\ \theta_1,\ \theta_2,\ \theta_3)$$

$$\therefore \qquad \delta R = uR_u+vR_v+wR_w+\theta_1R_1+\theta_2R_2+\theta_3R_3.$$

If the kite is strictly a plane surface with an axis of symmetry, the value of R cannot be affected by any reversal of u, θ_2, θ_3;

hence $\qquad R_u = R_2 = R_3 = 0$

where R_u, R_1 etc., denote the values of $\dfrac{\partial R}{\partial u}, \dfrac{\partial R}{\partial \theta_1}$ etc., in equilibrium.

Thus $$\delta R = vR_v+wR_w+\theta_1R_1 \qquad \ldots \quad \ldots \quad (1)$$

The values of R_v, R_w, R_1, will of course depend on the actual law of resistance assumed, and it is usual to assume that the resistance is proportional to the square of the velocity and also a function of the angle of attack.

Let us assume then

$$R = KSV^2 f(\alpha)$$

K and S being constants and when v, w, θ_1 are impressed the component velocities of any element S, whose distance from the x-axis is y, become v_0+v and $w_0+w+y\theta_1$.

$$\therefore\ R_0+\delta R=KS[V^2+2vv_0+2w_0(w+y\theta_1)][f(\alpha)+\delta\alpha f'(\alpha)],$$

also $$\tan\alpha=\frac{w_0}{v}.\quad \therefore\ \sec^2\alpha\,\delta\alpha=\frac{v_0\delta w_0-w_0\delta v_0}{v_0^2}.$$

Now remembering that $v_0=V\cos\alpha$, $w_0=V\sin\alpha$, $\delta w_0=w+y\theta_1$, $\delta v_0=v$, we have

$$\delta\alpha=\frac{(w+y\theta_1)\cos\alpha-v\sin\alpha}{V};$$

and on equating coefficients with (1)

$$R_v=KSV\,[2f(\alpha)\cos\alpha-f'(\alpha)\sin\alpha]\ldots \qquad \ldots \qquad \ldots \quad \text{(i)}$$

$$R_w=KSV\,[2f(\alpha)\sin\alpha+f'(\alpha)\cos\alpha] \qquad \ldots \qquad \ldots \quad \text{(ii)}$$

$$R_1=KSV\,[2f(\alpha)\sin\alpha+f'(\alpha)\cos\alpha]\,\bar{y} \qquad \ldots \qquad \ldots \quad \text{(iii)}$$

where $\bar{y}=\Sigma(yS)$.

To find the change in ηR. We assume

$$\eta=c'\phi(\alpha)$$

where c is a quantity depending on the linear dimensions of the kite. We have also

$$\eta R=\int\eta dR,$$

where η is the C. P. of an element dS, the thrust on which is dR or $KV^2f(\alpha)dS$.

When additional velocity components are impressed, α becomes $\alpha+\delta\alpha$ and V^2 becomes

$$(V\cos\alpha+v)^2+(V\sin\alpha+w+y\theta_1)^2;$$

hence neglecting squares and products of small quantities

$$\eta_0\delta R+R_0\delta\eta=\eta_0(vR_v+wR_w+\theta_1R_1)$$
$$+\frac{R_0c'\phi'(\alpha)}{V}(w\cos\alpha-v\sin\alpha+\bar{y}\theta_1\cos\alpha).$$

We have also from Euler's geometrical equations, since θ is small,

$$\theta_1=\dot{\phi}=\dot{\epsilon}$$

When these values are substituted we shall have six linear differential equations of the first order to determine the motion completely.

4. It will now be noticed that the equations (1)...(6) of the previous article are separable into two groups, the system (2), (3), (4) determine the oscillations of the kite in the $y\,z$ plane, while (1), (5), (6) determine the sideways rotations and translations.

Equations of symmetric oscillations: When the above values of δR and $\delta\eta$ are substituted the system (2), (3), (4) become

$$M\dot{v}=\epsilon(T\sin\phi_0+\chi+W\sin\phi_0)$$
$$M\dot{w}=-vR_v-wR_w-\theta_1R_1$$
$$A\dot{\theta}_1=-\eta_0(vR_v+wR_w+\theta_1R_1)$$
$$-\frac{R_0c'\phi'(\alpha)}{V}(w\cos\alpha-v\sin\alpha+\bar{y}\theta_1\cos\alpha)$$

and to solve this system we assume as usual v, w, ϵ proportional to $e^{\lambda t}$ and eliminate the variables v, w, ϵ. The determinantal equation in λ is reduced to

$$a\lambda^4+b\lambda^3+c\lambda^2+d\lambda+e=0\,;$$

since $\quad \mathrm{T}\sin\overline{\phi_0+\chi}+\mathrm{W}\sin\phi_0=\mathrm{R}_0$

$$a=\mathrm{AM}^2$$

$$b=\mathrm{M}^2\left(\mathrm{R}_0\frac{c'\phi'(\alpha)}{\mathrm{V}}\bar{y}\cos\alpha+\eta_0\mathrm{R}_1\right)+\mathrm{AMR}_w$$

$$c=\mathrm{M}\frac{c'\phi'(\alpha)}{\mathrm{V}}\mathrm{R}_0\cos\alpha(\bar{y}\mathrm{R}_w-\mathrm{R}_1)$$

$$d=\mathrm{M}\left(\eta_0\mathrm{R}_0\mathrm{R}_v-\mathrm{R}_0^2\frac{c'\phi'(\alpha)}{\mathrm{V}}\sin\alpha\right)$$

$$e=-\frac{\mathrm{R}_0^2}{\mathrm{V}}c'\phi'(\alpha)(\mathrm{R}_w\sin\alpha+\mathrm{R}_v\cos\alpha).$$

Asymmetric Oscillations: Here we have the three equations.

$$\mathrm{M}\dot{u}=\mathrm{T}l+\mathrm{W}\theta=\theta(\mathrm{W}-\mathrm{T}\cos\chi)-\psi\,\mathrm{T}\sin\chi$$

$$\mathrm{B}\dot{\theta}_2=-f\mathrm{T}n=-h_0\mathrm{T}\sin\overline{\phi_0+\chi}\,(\theta\cos\phi_0-\psi\sin\phi_0)$$

$$\mathrm{C}\dot{\theta}_3=\mathrm{T}(fm-hl)=-h_0\mathrm{T}\cos\overline{\phi_0+\chi}\,(\theta\cos\phi_0-\psi\sin\phi_0)\,;$$

we have also from Euler's geometrical equations, since θ is small,

$$\dot{\theta}=\theta_2\sin\phi_0+\theta_3\cos\phi_0$$

$$\dot{\psi}=\theta_2\cos\phi_0-\theta_3\sin\phi_0.$$

If we differentiate the above three equations, and substitute these values of $\dot{\theta}$ and $\dot{\psi}$, we get on slight simplification

$$\mathrm{M}\ddot{u}=\theta_2(\mathrm{W}\sin\phi_0-\mathrm{T}\sin\overline{\phi_0+\chi})+\theta_3(\mathrm{W}\cos\phi-\mathrm{T}\cos\overline{\phi_0+\chi})$$

$$=p_2\theta_2+p_3\theta_3 \text{ say}$$

$$\mathrm{B}\ddot{\theta}_2=h_0\mathrm{T}\sin\overline{\phi_0+\chi}\cdot\theta_3$$

$$\mathrm{C}\ddot{\theta}_3=h_0\mathrm{T}\sin\overline{\phi_0+\chi}.\theta_2\,;$$

and to solve this system, we assume as before u, θ_2, θ_3, proportional to $e^{\lambda t}$, and the determinantal equation in λ is

$$\begin{vmatrix} \mathrm{M}\lambda^2 & -p_2 & -p_3 \\ 0 & \mathrm{B}\lambda^2 & h_0\mathrm{T}\sin\overline{\phi_0+\chi} \\ 0 & h_0\mathrm{T}\sin\overline{\phi_0+\chi} & -\mathrm{C}\lambda^2 \end{vmatrix}=0,$$

i.e. $\quad \lambda^2(\mathrm{BC}\lambda^4+h_0^2\mathrm{T}^2\sin^2\overline{\phi_0+\chi})=0.$

5. *Conditions of Stability.* The dependence of stability on the roots of these equations will be found fully discussed in Routh, vol. II, chap. vi. For stability, the real roots or the real part of the imaginary roots

must be negative, and this will be the case, all the coefficients being finite, if a, b, c, d, e and $(bcd-eb^2-ad^2)$ have the same sign.

It follows, since a is positive, that all the coefficients should be positive.

First of all, we notice that if in the equations for symmetric oscillations, we neglect the displacement of the centre of pressure, and the position of equilibrium be such that the string passes through the centre of gravity, then all the coefficients in the biquadratic with the exception of the first two vanish, indicating lack of longitudinal stability. This result has an important application to aeroplanes.†

If the displacement of the centre of pressure be not neglected, then we have, since $R_w \sin \alpha + R_v \cos \alpha = 2\ KSVf(\alpha)$,

$$e = -\frac{R_o^2}{V} c\phi'(\alpha).2KSVf(\alpha)$$

which shows that for stability $\phi'(\alpha)$ must be negative, or the centre of pressure must move forward as the angle of attack diminishes. This condition is satisfied in practice.

With regard to lateral stability, we have obtained the equations on the assumption that the resultant tension does not intersect the axis of symmetry. But if the string simply bifurcates into two branches, then it can be easily proved from considerations of elementary statics that the string always intersects the axis of symmetry.

Let H be the point of bifurcation and E the point where the string meets the ground. Let the two points of attachment to the kite be C and D. Now if the direction of the string be fixed in space, then H is a fixed point, and the three forces acting on it are the three tensions HC, HD, and HE; and for the equilibrium of H these must lie in one plane, or EH produced must intersect CD.

In this case $M=0$ and $N=-hTl$; and it can be easily proved in the same manner as before that the coresponding equation in λ is

$$M\lambda^2\left(\lambda^2+\frac{h_0 W \sin \alpha}{C}\right)=0,$$

a biquadratic with two zero roots.

We conclude, in any case, that in strictly plane kites lateral stability is impossible so long as the centre of pressure does not deviate

† See: *Stability in Aviation* by G. H. Bryan, F. R. S. (Macmillan)

from the axis of symmetry‡. There are several other facts which yet remain to be explained, namely the peculiar behaviour of the kite when the wind velocity changes and the necessity of adjusting the two lengths HC and HD properly. To discuss these cases it will be necessary to take into account the variation in the magnitude and the direction of the tension. The next paper will be devoted to the discussion of the case of longitudinal stability with finite string, and explanation of some well-known facts connected with kite-flying.

I have to thank Prof. Bryan and Mr. Berwick for many valuable suggestions.

(*To be continued*).

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‡ In practice the lateral stability is increased by the bending of the plane of the kite under wind pressure.

SHORT NOTES.

The Mathematical Association of America.

On December 30 and 31, 1915, there was held at Columbus, Ohio, the organisation meeting of a new national mathematical association, the call for which had been signed by 450 persons representing every state in the Union, the District of Columbia, and Canada. The object of the new Association is to assist in promoting the interests of mathematics in America, especially in the collegiate field. It is not intended to be a rival of any existing organisation, but rather to supplement the Secondary Associations on the one hand, and the American Mathematical Society on the other, the former being well organized and effective in their field, and the latter having definitely limited itself to the field of scientific research. In the field of collegiate mathematics, however, there has been, up to this time, no organization and no medium of communication among the teachers, except the American Mathematical Monthly, which for the past three years has been devoted to this cause. The new organization, which has been named the Mathematical Association of America, has taken over the Monthly as its Official Journal.

There were 104 persons present at the organization meeting. The constitution and by-laws together with a full report of the proceedings will be published in the January issue of the Monthly. The following officers were elected :

President, Professor E. B. Hedrick, University of Missouri.

First Vice-President, Professor E. V. Huntington, Harvard University.

Second Vice-President, Professor G. A. Miller, University of Illinois.

Secretary-Treasurer, Professor W. D. Cairns, Oberlin College.

Publication Committee, Professor H. E. Slaught, University of Chicago, Managing Editor, Professor W. H. Bussey, University of Minnesota, and Professor R. D. Carmichael, University of Illinois.

These officers, together with the following, constitute the Executive Council :

Professor R. C. Archibald, Brown University.

Do. Florian Cajori, Colorado College.

Do. B. F. Finkel, Drury College.

Do. D. N. Lehmer, University of California.

Professor E. H. Moore, University of Chicago.
Do. R. E. Moritz, University of Washington.
Do. M. B. Porter, University of Texas.
Do. K. D. Swartzel, Ohio State University.
Do. J. N. Van der Vries, University of Kansas.
Do. Oswald Veblen, Princeton University.
Do. J. W. Young, Dartmouth College.
Do. Alexander Ziwet, University of Wisconsin.

Note on Partial Fractions.

1. *Lemma.*—Suppose λ is a root of the equation

$$\phi(x)\equiv a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n=0;$$

then any fraction $f(\lambda)/F(\lambda)$ can be reduced to an integral expression of the $(n-1)^{th}$ degree in λ.

This is known as Tschirnhaussen's transformation and is explained in books on *Theory of Equations.* The following method of proof may be of interest.

By the method of continued fractions the fraction $\phi(\lambda)/F(\lambda)$ can be written in the form

$$A_0+\frac{1}{A_1+}\,\frac{1}{A_2+}\cdots$$

where the A's are functions of λ. If the penultimate convergent to the continued fraction be P/Q, we have

$$P.F(\lambda)-Q\cdot\phi(\lambda)=\pm 1;$$

so that $$P.F(\lambda)=\pm 1,\text{ or } f(\lambda)/F(\lambda)=\pm f(\lambda).P,$$

which is an integral function. Evidently, this integral function can be written

$$\phi(\lambda).Q'+R$$

by division by $\phi(\lambda)$, where R is of the $(n-1)$th degree in λ.

Hence, the given fraction

$$f(\lambda)/F(\lambda)=\phi(\lambda)\cdot Q'+R=R,$$

which proves the Lemma.

2, To reduce any expression

$$\frac{\psi(x)}{\Pi\,[\phi(x)]}$$

where the denominator consists of different real integral factors $\phi(x)$, we proceed thus:

Put $$\frac{\psi(x)}{\Pi\,[\phi(x)]}=\Sigma\left[\frac{\phi_1(x)}{\phi(x)}\right]; \quad \ldots \quad \ldots \quad (1)$$

and multiply both sides by $\phi(x)$ and afterwards write λ for x, where λ is a root of $\phi(x)=0$.

Then by the foregoing Lemma, the left hand side can be reduced to an integral function R of a degree lower than $\phi(x)$; and the right hand side will be $\phi_1(\lambda)$.

Hence $$R=\phi_1(\lambda);$$

In other words, *the numerator* $\phi_1(x)$ *of the partial fraction corresponding to* $\phi(x)$ *is the transform of the given fraction after multiplication by* $\phi(x)$.

3. As an example consider the case of the cubic

$$\phi(x)\equiv x^3+px^2+qx+r,$$

and the corresponding numerator Px^2+Qx+R.

Write $f(x)$ for the left hand member of (1) multiplied by $\phi(x)$. Then the values of P, Q, R may also be directly obtained by the following artifice.

Let α, β, γ be the roots of $x^3+px^2+qx+r=0$, so that

$$P\alpha^2+Q\alpha+R-f(\alpha)=0,$$
$$P\beta^2+Q\beta+R-f(\beta)=0,$$
$$P\gamma^2+Q\gamma+R-f(\gamma)=0;$$

solving these for P, Q, R, we get

$$\frac{P}{\begin{vmatrix} 1 & \alpha & f(\alpha) \\ 1 & \beta & f(\beta) \\ 1 & \gamma & f(\gamma) \end{vmatrix}}=\frac{Q}{\begin{vmatrix} 1 & \alpha^2 & f(\alpha) \\ 1 & \beta^2 & f(\beta) \\ 1 & \gamma^2 & f(\gamma) \end{vmatrix}}=\frac{R}{\begin{vmatrix} \alpha & \alpha^2 & f(\alpha) \\ \beta & \beta^2 & f(\beta) \\ \gamma & \gamma^2 & f(\gamma) \end{vmatrix}}=\frac{-1}{\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}}.$$

The first three determinants are divisible by the last, and the quotients being symmetric functions of α, β, γ are readily expressed in terms of p, q, r. Any of the first three determinants can be expanded by Maclaurin's theorem and the result expressed in a series thus:

$$\begin{vmatrix} 1 & \alpha & f(\alpha) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}=\Sigma(D_r),$$

where $D_r=\dfrac{f^r(o)}{r!}\begin{vmatrix} 1 & \alpha & \alpha^r \\ 1 & \beta & \beta^r \\ 1 & \gamma & \gamma^r \end{vmatrix}=S_{r-2}\dfrac{f^r(o)}{r!}\times\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}$

Hence $$P=-\Sigma\left[\frac{f^r(o)}{r!}S_{r-2}\right]$$

where S_{r-2}=sum of homogeneous products of α, β, γ of $(r-2)$ dimensions.

M. T. NARANIENGAR.

Newton's Theorems.

1. Let $S_r \equiv a\alpha^r + b\,\beta^r + c\,\gamma^r + \ldots$
where $\alpha, \beta, \gamma \ldots$ are all the roots of the equation

$$f(x) \equiv x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots\ldots (-1)^n p_n = 0, \quad \ldots \quad \ldots \quad (1)$$

and a, b, c are any given numbers.

Multiply (1) by x^{m-n}, and substitute $\alpha, \beta, \gamma \ldots$ for x in succession; multiply the results by $a, b, c \ldots$ in order and add. We have

$$s_m - p_1\, s_{m-1} + p_2\, s_{m-2} \ldots\ldots (-1)^n\, s_{m-n} = 0 \; [m > n] \quad \ldots \quad \ldots \quad (2)$$

which is an extension of Newton's first theorem.

2. When $m < n$, we find

$$s_m - p_1\, s_{m-1} + p_2\, s_{m-2} \ldots - (-1)^m (p_m\, s_o - q_m) = 0, \quad \ldots \quad (3)$$

where q_m denotes the sum of products of the roots with coefficients equal to the sums of $a, b, c \ldots$ corresponding to the roots not occurring in the product.

For, by the above process,

$$s_{n-1} - p_1\, s_{n-2} + p_2\, s_{n-3} - \ldots\ldots (-1)^n p_n s_{-1} = 0.$$

But $\quad p_n s_{-1} = \alpha\beta\gamma \ldots (a\alpha^{-1} + b\beta^{-1} + c\gamma^{-1} + \ldots)$

$$= a\beta\gamma\delta \ldots + b\alpha\gamma \ldots + \ldots = q_{n-1}.$$

Similarly

$$s_{n-2} - p_1 s_{n-3} + \ldots (-1)^{n-2} \{ p_{n-2} s_o - p_{n-1} s_{-1} + p_n s_{-2} \} = 0,$$

and

$$p_{n-1} s_{-1} - p_n s_{-2} = \alpha\beta\gamma \ldots \left\{ \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \ldots \right) \left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \ldots \right) - \left(\frac{a}{\alpha^2} + \frac{b}{\beta^2} + \frac{c}{\gamma^2} + \ldots \right) \right\}$$

$$= \alpha\beta\gamma \ldots \left(\frac{a+b}{\alpha\beta} + \frac{b+c}{\beta\gamma} + \ldots \right)$$

$$= q_{n-2};$$

and so on.

3. Again, suppose

$$\phi(\alpha) = a_o \alpha^m + a_1 \alpha^{m-1} + \ldots + a_m;$$

then writing

$$s_r = \phi^r(\alpha) + \phi^r(\beta) + \phi^r(\gamma) + \ldots,$$

we obtain

$$q_o\, s_m + q_1\, s_{m-1} + \ldots m\, q_m = 0, \; [m < n] \quad \ldots \quad (4)$$

$$q_o\, s_m + q_1\, s_{m-1} + \ldots q_n\, s_{m-n} = 0, \; [m > n] \quad \ldots \quad (5)$$

where the q's are the co-efficients of the equation whose roots are $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$, etc.

4. *Examples:*

(i) Let $s_1 = \Sigma(\alpha^p)$, $s_2 = \Sigma(\alpha^{2p}), \ldots$, then $q_o, q_1 \ldots$ are the co-efficients of the rationalised equation $f\left(y^{1/p}\right) = 0$.

(ii) Suppose $f(x)=(x-\alpha)^a.(x-\beta)^b.(x-\gamma)^c....$
$$=x^n+P_1x^{n-1}+P_2x^{n-2}+...P_n.$$

Then $\quad f'(x)/f(x)=a/(x-\alpha)+b/(x-\beta)+c/(x-\gamma)+...$
$$=\Sigma\{s_r/x^{r+1}\},$$

where $s_r=a\alpha^r+b\beta^r+c\gamma^r+.. ...$

Proceeding as in Cajori's *Theory of Equations*, § 68, we get

$$\left.\begin{array}{l} s_1+n\,P_1=(n-1)\,P_1.........\\ s_2+P_1\,s_1+n\cdot P_2=(n-2)P_2\\ ...\quad ...\quad ...\end{array}\right\}\quad\therefore\quad \begin{array}{r} s_1+P_1=0,\\ s_2+P_1\,s_1+2P_2=0,\\ ...\quad ...\quad ...\end{array}$$

where

$n=a+b+c...\,;-P_1=a\alpha+b\beta+c\gamma+...=s_1;$

P_2=sum of the products of roots two at a time

$$=ab\alpha\beta+bc\beta\gamma+...+\frac{a(a-1)}{1.2}\alpha^2+\frac{b(b-1)}{1.2}\beta^2+\frac{c(c-1)}{1.2}\gamma^2+...$$

$$=\tfrac{1}{2}\{(a\alpha+b\beta+...)^2-(a\alpha^2+b\beta^2+...)\}=\tfrac{1}{2}(s_1^2-s_2):\ \&c., \&c.$$

Hence also we can derive the result (3) above.

(iii) To find the identical relations satisfied by $s_1, s_2,...$, we write down $(n+1)$ relations for particular values of m in (2) or (3) and eliminate the p's. Thus, in the case of the cubic

$$x^3-p_1x^2+p_2x-p_3=0,$$

we may take

$$\begin{array}{l} s_3-p_1s_2+p_2s_1-p_3s_0=0\\ s_4-p_1s_3+p_2s_2-p_3s_1=0\\ s_5-p_1s_4+p_2s_3-p_3s_2=0\\ s_6-p_1s_5+p_2s_4-p_3s_3=0\end{array}$$

and eliminate p_1, p_2, p_3 to obtain the identical relation

$$\begin{vmatrix} s_3 & s_2 & s_1 & s_0\\ ... & ... & ... & ...\\ ... & ... & ... & ...\\ ... & ... & ... & ...\end{vmatrix}=0.$$

M. T. NARANIENGAR.

The Face of the Sky for May and June

The Sun

enters Gemini on May 21 and the Summer solstice on June 21 at 11-54 P.M.

The Moon

	May. D.	May. H.	May. M.	June. D.	June. H.	June. M.
New Moon ...	2	10	59	1	1	7
First Quarter ...	10	2	17	9	5	29
Full Moon ...	17	7	41	16	3	12
Last Quarter ...	24	10	46	22	6	46
New Moon ...		...		... 30	4	13

The Planets.

Mercury attains its greatest elongation (E) on May 12. It is stationary on May 25 and is in inferior conjunction on June 6. It is stationary on June 18 and attains its greatest elongation 21° 50′ (W) on June 30. It is in conjunction with the Moon on May 4, June 1 and June 28.

Venus continues an evening star. It attains its greatest brilliancy on June 1. It is stationary on June 12. It is in conjunction with the Moon on May 6 and on June 4 and with Saturn on May 24 and June 22.

Mars is in quadrature to the Sun on May 15. It is in conjunction with the Moon on May 10 and June 8.

Jupiter is in conjunction with the Moon on May 28, and on June 25 at 5-11 A. M.

Saturn is in conjunction with the Moon on May 7 and June 3 at 11-48 P.M. and with 8 Geminorum on June 4.

Uranus is in quadrature to the Sun on May 10. It is stationary on May 25. It is in conjunction with the Moon on May 23 and June 19.

Neptune is in conjunction with the Moon on May 8 and June 5.

V. RAMESAM.

SOLUTIONS.

Questions 586 & 628.

(K. V. Anantanarayana Sastri, B.A.) :—Prove that the length of the first negative pedal of the loop of the Folium of Descartes $x^3+y^3-3\,axy=0$, is $6a-a\,[\pi-\sqrt{2}\ \log(1+\sqrt{2})]$.

Solution by T. P. Krishnaswami.

It is known that the negative pedal of any curve is the envelope of a line perpendicular to the radius vector through its other extremity. In the present case the line is

$$x\cos\alpha+y\sin\alpha=\frac{3\,at\,\sqrt{1+t^2}}{1+t^3},\ \text{where } t=\tan\alpha$$

$$i.e.\quad x\cos\alpha+y\sin\alpha=\frac{3\,a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\quad\ldots\quad\ldots\quad\ldots\quad(1)$$

Again, it is also known that the radius of curvature of the envelope of of the line

$$x\cos\alpha+y\sin\alpha=f(\alpha)\quad\ldots\quad\ldots\quad\ldots\quad(2)$$

is $f(\alpha)+f''(\alpha)$. : (See Edward's *Diffl. Calc.*, P. 309 Ex. 17) and, as α and ψ differ only by a constant quantity, $d\alpha=d\psi$ and therefore we get from (2) that the arc

$$s=\int[f(\alpha)+f''(\alpha)]\,d\alpha$$

$$=\int_{\frac{1}{2}\pi}^{0}\frac{3a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\,d\alpha+\left[\frac{d}{d\alpha}\left(\frac{3a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\right)\right]_{\frac{1}{2}\pi}^{0}$$

$$=\int_{\frac{1}{2}\pi}^{0}\frac{3a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\,d\alpha+6a.$$

Now, since $y=x$ is a line of symmetry, the integral on the right side can be written

$$2a\int_0^{\frac{1}{4}\pi}\left\{\frac{1}{\sin\alpha+\cos\alpha}-\frac{\sin\alpha+\cos\alpha}{1-\sin\alpha\cos\alpha}\right\}d\alpha$$

$$=a\sqrt{2}\int_0^{\frac{\pi}{4}}\frac{d\alpha}{\cos\left(\alpha-\frac{\pi}{4}\right)}\,4-a\sqrt{2}\int_0^{\frac{1}{4}\pi}\frac{\cos\left(\alpha-\frac{\pi}{4}\right).}{2-\cos 2\left(\alpha-\frac{\pi}{4}\right)}$$

$$=a\sqrt{2}\left[\log\tan\left(\frac{\pi}{4}+\frac{\pi}{2}-\frac{\pi}{8}\right)\right]_0^{\frac{1}{4}\pi}$$

$$-4a\sqrt{2}\int_0^{\frac{\pi}{4}} \frac{d\sin\left(\alpha-\frac{\pi}{4}\right)}{1+2\sin^2\left(\alpha-\frac{\pi}{4}\right)}$$

$$=-a\sqrt{2}\log\left(\tan\frac{\pi}{8}\right)-4a\tan^{-1}\left\{\sqrt{2}\sin\left(\alpha-\frac{\pi}{4}\right)\right\}$$

$$=a\sqrt{2}\log(\sqrt{2}+1)-a\pi.$$

Hence $s=6a-a\pi+a\sqrt{2}\log(\sqrt{2}+1)$.

Question 630.

(S. NARAYANA AIYAR, M. A) :—If

$$A_n=\frac{(a-b)(a-bx)\ldots\ldots\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)\ldots\ldots\ldots\ldots(1-x^n)},$$

and

$$B_n=\frac{(b-a)(b-ax)\ldots\ldots\ldots\ldots(b-ax^{n-1})}{(1-x)(1-x^2)\ldots\ldots\ldots\ldots(1-x^n)},$$

shew that

$$(-1)^n B_n=\begin{vmatrix} A_1, & 1, & 0, & 0 & \ldots & \ldots & \ldots & 0 \\ A_2, & A_1, & 1, & 0 & \ldots & \ldots & \ldots & 0 \\ A_3, & A_2, & A_1, & 1 & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n-1}, & A_{n-2}, & A_{n-3}, & A_{n-4} & \ldots & \ldots & \ldots & 1 \\ A_n, & A_{n-1}, & A_{n-2} & A_{n-3}, & \ldots & \ldots & \ldots & A_1 \end{vmatrix}$$

with a similar determinant for $(-1)^n A_n$ in terms of the constituents $B_1, B_2, \ldots\ldots B_n$.

Solution by N. Sankara Aiyar, M. A.

The given determinant is equal to

$$A_1\begin{vmatrix} A_1 & 1 & 0 & \ldots & 0 \\ A_2 & A_1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n-1} & A_{n-2} & \ldots & \ldots & A_1 \end{vmatrix} - \begin{vmatrix} A_2 & 1 & 0 & \ldots & 0 \\ A_3 & A_1 & 1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & & \\ A_n & A_{n-2} & \ldots & \ldots & A_1 \end{vmatrix}$$

We shall prove the result by Mathematical Induction.

We should prove that

$$\begin{vmatrix} A_2 & 1 & 0 & \dots & 0 \\ A_3 & A_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_n & A_{n-2} & \dots & \dots & \dots \end{vmatrix} = A_1(-1)^{n-1}B_{n-1} - (-1)^n B_n$$

$$= (-1)^{n-1}B_{n-1}\left(\frac{a-b}{1-x} + \frac{b-ax^{n-1}}{1-x^n}\right)$$

$$= (-1)^{n-1}B_{n-1}\frac{(a-bx)(1-x^{n-1})}{(1-x)(1-x^n)};$$

that is,

$$(-1)^{n-2}B_{n-2}A_1 - \begin{vmatrix} A_3 & 1 & \dots & 0 \\ A_4 & A_1 & \dots & 0 \\ A_n & A_{n-3} & \dots & A_1 \end{vmatrix} = (-1)^{n-1}B_{n-1}\frac{a-bx}{(1-x)}\frac{1-x^{n-1}}{1-x^n}$$

$$\therefore \quad \begin{vmatrix} A_3 & 1 & \dots & \dots & 0 \\ A_4 & A_1 & \dots & \dots & 0 \\ \dots & \dots & \dots \\ A_n & A_{n-3} & \dots & A_1 \end{vmatrix} = (-1)^{n-2}B_{n-2}\frac{a-bx}{1-x}\left(\frac{a-b}{1-x^2} + \frac{b-ax^{n-2}}{1-x^n}\right)$$

$$= (-1)^{n-2}B_{n-2}.\frac{a-bx}{1-x}.\frac{a-bx^2}{(1-x^2)}.\frac{1-x^{n-2}}{1-x^n};$$

and so on, until, finally, we have

$$\begin{vmatrix} A_{n-1} & 1 \\ A_n & A_1 \end{vmatrix} = (-1)^2 B_2 \frac{a-bx}{1-x}\dots\frac{a-bx^{n-2}}{1-x^{n-2}}.\frac{1-x^2}{1-x^n}.$$

But $\qquad A_{n-1}A_1 - A_n = A_{n-1}\left(\frac{a-b}{1-a} - \frac{a-bx^{n-1}}{1-x^n}.\right)$

$$= -A_{n-1}\frac{b-ax}{1-x}.\frac{1-x^{n-1}}{1-x^n}.$$

$$= \frac{b-ax}{1-x}.\frac{b-a}{1-x}.\frac{a-bx}{1-x^2}\dots\frac{a-bx^{n-2}}{1-x^{n-1}}.\frac{1-x^{n-1}}{1-x^n}.$$

Hence we should get

$$\frac{b-a}{1-x}.\frac{b-ax}{1-x^2}.\frac{a-bx}{1-x}\dots\frac{a-bx^{n-2}}{1-x^{n-1}}.\frac{1-x^2}{1-x^n}$$

$$= \frac{b-a}{1-x}.\frac{b-ax}{1-x}.\frac{a-bx}{1-x^2}\dots\frac{a-bx^{n-2}}{1-x^{n-1}}.\frac{1-x^{n-1}}{1-x^n},$$

which is obvious.

Thus, if the formula is true up to $n-1$, it is true up to n. Hence, &c. Similarly the case of A_n in terms of the B's can be proved.

Question 639.

(V. V. S. NARAYAN) :—The polar circles of the five triangles' external to a pentagon, which are formed by producing its sides have a common orthogonal circle.

Solution by K. B. Madhava.

Let P be the conic touching the sides of the pentagon A B C D E, and Q its director circle. If the $\triangle$ external to the pentagon formed by producing the sides A E & B C be called $\triangle_1$; and the polar circle of this triangle R_1, and so on; then, by Gaskin's Theorem (Durell, Theorem 215; or Salmon, § 375, Ex 2)

R_1 is orthogonal to Q.

Similarly all the circles $R_2 \ldots R_5$ are orthogonal to Q. Hence the result.

Question 641.

(SELECTED):— " There cannot be five prime numbers in arithmetical progression unless their common difference be divisible by $2\times3\times5=30$ except when the first term of the progression is 5 ".

[Barlow's *Theory of Numbers*, p, 65.]

Illustrate or criticise the above statement.

Solution by N. Sankara Aiyar, M.A., and R. D. Karve.

Let the five prime numbers be

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d.$$

First, a and d are prime to each other and $a \not\equiv 3$; for otherwise $a+3d$ would not be prime; a may be 5 and then $d \not\equiv 0$ (mod 5).

If a is greater than 5 then one of the four numbers

$$a+d,\ a+2d,\ a+3d,\ a+4d$$

will be $\equiv 0$ (mod 5), unless $d \equiv 0$(mod 5).

Similarly, one of $a+d$, $a+2d$, $a+3d$ will be $\equiv 0$ (mod 3), unless $d \equiv 0$ (mod 3); and so also $d \equiv 0$ (mod 2).

Hence $d \equiv 0$ (mod 30), unless $a=5$, in which case $d \equiv 0$ (mod 6).

Examples :—(1) 5, 11, 17, 23, 29;
(2) 5, 17, 29, 41, 53;
(3) 5, 47, 89, 131, 173;
(4) 5, 53, 101, 149, 197;
(5) 71, 101, 131, 161, 191;
(6) 71, 131, 191, 251, 311;
(7) 73, 223, 373, 523, 673;
(8) 89, 599, 1109, 1619, 2129.

It must however be noted that if $d \not\equiv 5$, there is the possibility of a sixth number of the A.P. being prime. Thus if $a=7$, $d \not\equiv 0 \pmod 7$ $a \not\equiv 7$ we have

$$\begin{array}{l} 7, 37, 67, 97, 127, 157; \\ 7, 157, 307, 457, 607, 757; \\ 11, 71, 131, 191, 251, 311; \\ 11, 491, 971, 1451, 1931, 2411. \end{array}$$

But there will not be seven such numbers, unless either $a=7$ or $d=0$ (mod 210). Thus in the second set above 907 also is prime.

Question 649.

(S. Malhari Rao, B.A.) :—A gentleman has one third of an acre of vacant land round his house. He wishes to divide it into five different plots to grow five different kinds of flowers. What must be the areas of these plots so that they may be in A. P., and each of them may be contained an exact number of times in an acre? Shew that there cannot be more than one set of values for the areas.

Solution by V. Anantaraman and H. K. Chakrabarty, B.Sc.

Let the unit area be an acre. Then, since the area of the five plots is $\frac{1}{3}$ and they are in A. P., the area of the middle plot should be $\frac{1}{15}$.

Suppose the second and fourth plots are $\frac{1}{x}, \frac{1}{y}$; then by the question

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{15},$$

i.e.
$$15(x+y)=2xy,$$

where x and y are positive integers.

Now xy must be divisible by 15, and the following two distinct cases arise:

(i) $x=15a$; so that $15a+y=2ay$, whence $y=ab$ and $15+b=2ab$, resulting in

$$\begin{array}{l} b=1,\ a=8,\ y=8,\ x=120 \\ b=3,\ a=3,\ y=9,\ x=45 \\ b=5,\ a=2,\ y=10,\ x=30 \\ b=15,\ a=1,\ y=15,\ x=15. \end{array}$$

(ii) $\left.\begin{array}{l} x=5a \\ y=3b \end{array}\right\}$, so that $5a+3b=2ab$, whence $5a/b$ is an integer, resulting in $b=5$, $a=3$, $x=y=15$;

or $\quad a=bt$, $5t+3=2bt$ and $t=1, b=4, a=4, x=20, y=12$;
$\quad t=3, b=3, a=9, x=45, y=9.$

Thus the possible values of x and y are

$$\left.\begin{array}{l} x=120,\ 45,\ 30,\ 20,\ 15 \\ y=\ \ 8,\ \ 9,\ 10,\ 12,\ 15. \end{array}\right\}$$

Also, since the middle plot is $\frac{1}{15}$, twice the common difference is less than $\frac{1}{15}$. That is

$$\frac{1}{y}-\frac{1}{x}<\frac{1}{15}.$$

Hence, the only admissible values of x and y are 20 and 12; and the singular solution is

$$\frac{1}{30}, \frac{1}{20}, \frac{1}{15}, \frac{1}{12}, \frac{1}{10}.$$

Question 651.

(S. Malhari Rao, B. A.):—The Nasik or pan-diagonal Magic square (*a*) has been constructed by placing the series of natural numbers from 1 up to 25 along the combined paths of a Knight and a Rook in chess. By adopting another method, and guided by the altered positions 1, 7, 13, 19 and 25 as shown in square (*b*), fill up the vacant cells in it so that it may be magic and pan-diagonal.

(*a*)

12	25	8	16	4
18	1	14	22	10
24	7	20	3	11
5	13	21	9	17
6	12	2	15	23

(*b*)

13				
	19			
		25		
			1	
				7

Solution by H. R. Chakrabarty, B. Sc.

The completed square is

13	10	2	24	16
22	19	11	8	5
6	3	25	17	14
20	12	9	1	23
4	21	18	15	7

Here the method adopted is the combined path of a Knight and a Bishop in chess. Knight's path is the main course, but when obstructed, Bishop's path is taken.

Question 652.

(S. P. SINGARAVELU MUDALIAR) :—From a point (eccentric angle ϕ) of an ellipse of semi-axes a, b three normals are drawn to the ellipse ; show that the square of the radius of the circle passing through the feet of the normals is

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

Solution (1) *by M. V. Arunachala Sastry, M.A.* (2) *by N. Sankara Aiyar, M.A.*

(1) The circle in question is evidently Joachimsthal's circle of the point ($a\cos\phi$, $b\sin\phi$). Its equation is given in page 219, equation (549), in Casey's Analytical Geometry.

If, in equation (549), we put $x'=a\cos\phi$, $y'=b\sin\phi$, and $u=a^2+b^2$, we get, as the required equation of the circle in question,

$$x^2+y^2-\frac{b^2}{a}x\cos\phi-\frac{a^2}{b}y\sin\phi=a^2+b^2.$$

[See Ex. 43, page 184, of Asquith's *Analytical Geometry.*]

Hence the square on the radius

$$=\left(a+\frac{b^2}{2a}\right)^2\cos\phi+\left(b+\frac{a^2}{2b}\right)\sin^2\phi.$$

(2) The equation of the circle through λ_1, λ_2, λ_3 is

$$x^2+y^2-\frac{a^2-b^2}{4aN}(l+m)x-\frac{a^2-b^2}{4ibN}(l-m)y$$
$$+\frac{(MN+L)(a^2-b^2)-2N(a^2+b^2)}{4N}=0$$

where $l=N^2+M$ and $m=LN+1$. (See. J.I.M.S., Q. 638.)

In the given case the equation giving the three values of λ is easily seen to be

$$(a^2-b^2)\cos\theta\cos\phi-a^2\sin\theta\cos\phi+b^2\cos\theta\sin\phi=0.$$

i.e. $$(a^2-b^2)\lambda^3\mu-\lambda^2(a^2+b^2)-\lambda(a^2+b^2)+a^2-b^2=0,$$

on simplification.

$$\therefore\qquad L=\frac{a^2+b^2}{a^2-b^2}\frac{1}{\mu},\ M=-\frac{a^2+b^2}{a^2-b^2},\ N=-\frac{1}{\mu}.$$

$$\therefore\ L=MN\,;\ \frac{(MN+L)(a^2-b^2)-2n(a^2+b^2)}{4N}=-(a^2+b^2).$$

$$\therefore\ N^2+M+LN+1=(N^2+1)(M+1),$$
$$N^2+M-LN-1=(N^2-1)(1-M).$$

$$\therefore\ N^2+M+LN+1=-\frac{\mu^2+1}{\mu^2}\,\frac{2b^2}{a^2-b^2}=-\frac{4b^2\cos\phi}{\mu(a^2-b^2)}$$

$$N^2+M-LN-1=\frac{\mu^2-1}{\mu^2}\,\frac{2a^2}{a^2-b^2}=-\frac{4a^2\sin\phi}{\mu(a^2-b^2)}.$$

The square on the radius

$$=\left\{\frac{a^2-b^2}{8aN}(l+m)\right\}^2+\left\{\frac{a^2+b^2}{8bN}(l-m)\right\}^2+a^2+b^2$$

$$=\left(\frac{b^2}{2a}\cos\phi\right)^2+\left\{\frac{a^2}{2b}\sin\phi\right\}^2+a^2+b^2$$

$$=a^2+b^2+\frac{b^4}{4a^2}\cos^2\phi+\frac{a^4}{4b^2}\sin^2\phi$$

$$=\left(a+\frac{b}{2a}\right)^2\cos^2\phi+\left(b+\frac{a}{2b}\right)^2\sin^2\phi.$$

Question 657.

(T. P. Thrivedi, M.A., LL.B.) :—Prove that the value of the determinant of the $(n+2)^{th}$ order

$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_1+a_2 & \dots & a_1+a_{n+1} \\ 1 & a_1+a_2 & 0 & \dots & a_2+a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1+a_{n+1} & a_2+a_{n+1} & \dots & 0 \end{vmatrix}$$

is equal to $-(-2)^n\,\Pi\, a_1\,\Sigma\,1/a_n)$. Also find the value of the determinant of the $(n+1)^{th}$ order obtained by deleting the first column and first row.

Solution by H. K. Chakrabarty, K. B. Madhava and A. Narasinga Rao.

Let the determinant $\begin{vmatrix} -1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 1 & \dots & 1 \\ 1 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & -1 \end{vmatrix}$

of the $(n+1)^{th}$ order be called Δ_{n+1} and the given determinant D_{n+2}, while that obtained by deleting the first column and first row of D_{n+2} is called D'_{n+1}. Then,

$$\Delta_{n+2}=\frac{1}{n-2}\begin{vmatrix} -2(n-1) & 1 & 1 & \dots & 1 \\ 0 & -1 & 1 & \dots & 1 \\ 0 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 1 & \dots & -1 \end{vmatrix}$$

obtained by subtracting all the other columns from $n-2$ times the first.

That is
$$\begin{aligned} \Delta_{n+1} &= -2\frac{n-1}{n-2}\Delta_n \\ \Delta_n &= -2\frac{n-2}{n-3}\Delta_{n-1} \\ \dots & \quad \dots \quad \dots \\ \Delta_4 &= -2\frac{2}{1}\Delta_3 \\ \Delta_3 &= (-2)^2. \end{aligned}$$

Hence multiplying by columns

$$\Delta_{n+1}=(-2)^n(n-1).$$

Now $$D_{n+2}=\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & -a_1 & a_2 & \dots & a_{n+1} \\ 1 & a_1 & -a_2 & \dots & a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & -a_{n+1} \end{vmatrix}$$

obtained by multiplying the 1st row of D_{n+2} by a_1, a_2 etc., and subtracting from the 2nd, 3rd etc., row respectively.

$$=\Pi a_1\begin{vmatrix} 0 & \frac{1}{a_1} & \frac{1}{a_2} & \dots & \frac{1}{a_{n+1}} \\ 1 & -1 & 1 & \dots & 1 \\ 1 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & -1 \end{vmatrix}$$

$$=\frac{\Pi a^1}{n-1}\begin{vmatrix} -\sum\frac{1}{a_1} & \frac{1}{a_1} & \frac{1}{a_2} & \cdots & \frac{1}{a_{n+1}} \\ 0 & -1 & 1 & \cdots & 1 \\ 0 & 1 & -1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & \cdots & -1 \end{vmatrix}$$

obtained by subtracting all the other columns from $n-1$ times the first.

$$=\frac{-\Pi a_1\Sigma\frac{1}{a_1}}{n-1}\cdot\Delta_{n+1}$$

$$=-(-2)^n\Pi a_1\sum\frac{1}{a_1}.$$

$$D'_{n+1}=\begin{vmatrix} 0 & a_1+a_2 & a_1+a_3 & \cdots & a_1+a_{n+1} \\ a_1+a_2 & 0 & a_2+a_3 & \cdots & a_2+a_{n+1} \\ a_1+a_3 & a_2+a_3 & 0 & \cdots & a_3+a_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1+a_{n+1} & a_2+a_{n+1} & a_3+a_{n+1} & \cdots & 0 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & a_1+a_2 & a_1+a_3 & \cdots & a_1+a_{n+1} \\ 0 & a_1+a_2 & 0 & a_2+a_3 & \cdots & a_2+a_{n+1} \\ 0 & a_1+a_3 & a_2+a_3 & 0 & \cdots & a_3+a_{n+1} \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_1+a_{n+1} & a_2+a_{n+1} & a_3+a_{n-1} & \cdots & 0 \end{vmatrix}$$

obtained by adding one row and one column more, making it a determinant of the $(n+2)^{th}$ order.

$$=-\frac{1}{2}\begin{vmatrix} (n-1)1+0 & 1 & 1 & \cdots & 1 \\ (n-1)a_1+\Sigma a_1 & 0 & a_1+a_2 & \cdots & a_1+a_{n+1} \\ (n-1)a_2+\Sigma a_1 & a_1+a_2 & 0 & \cdots & a_2+a_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (n-1)a_{n+1}+\Sigma a_1 & a_1+a_{n+1} & a_2+a_{n+1} & \cdots & 0 \end{vmatrix}$$

obtained by adding all the other columns to (-2) times the first

$$
= -\frac{n-1}{2}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & 0 & a_1+a_2 & \cdots & a_1+a_{n+1} \\ a_2 & a_1+a_2 & 0 & \cdots & a_2+a_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+1} & a_1+a_{n+1} & a_2+a_{n+1} & \cdots & 0 \end{vmatrix}
$$

$$
-\frac{\Sigma a}{2}\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & a_1+a_2 & \cdots & a_1+a_{n+1} \\ 1 & a_1+a_2 & 0 & \cdots & a_2+a_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_1+a_{n+1} & a_2+a_{n+1} & \cdots & 0 \end{vmatrix}
$$

obtained by splitting up into two determinants.

$$
= -\frac{n-1}{2}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -a_1 & a_2 & \cdots & a_{n+1} \\ 0 & a_1 & -a_2 & \cdots & a_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_1 & a_2 & \cdots & -a_{n+1} \end{vmatrix} - \frac{\Sigma a_1}{2} D_{n+2}
$$

$$
= -(n-)\frac{\Pi a_1}{2}\begin{vmatrix} -1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & \\ 1 & 1 & -1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} - \frac{\Sigma a_1}{2} D_{n+2}
$$

$$
= -(n-1)\frac{\Pi a_1}{2}\Delta_{n+1} - \frac{\Sigma a_1}{2} D_{n+2}
$$

$$
= -\frac{\Pi a_1}{2}(n-1)(-2)^n(n-1) + \frac{\Sigma a_1}{2}(-2)^n \Pi a_1 \Sigma\frac{1}{a_1}
$$

$$
= \Pi a_1 (-2)^{n-1}(n-1)^2 - \Pi a_1 \Sigma a_1 \Sigma\frac{1}{a_1}(-2)^{n-1}
$$

$$
= (-2)^{n-1}\Pi a_1\left[(n-1)^2 - \Sigma a_1 \Sigma\frac{1}{a_1}\right].
$$

Question 658.

(K. J. Sanjana, M.A.) :—Prove the identity

$$2[ax+by+(a+b)(x+y)]^3-6[ax+by+(a+b)(x+y)]$$
$$[a^2+ab+b^2]\,[x^2+xy+y^2]-$$
$$27\,ab\,xy\,(a+b)\,(x+y)=(a-b)\,(x-y)\,(2a+b)\,(2b+a)\,(2x+y)(2y+x),$$

and deduce the theorem of Cayley, that

$$ax+by+cz,\ bx+cy+az,\ cx+ay+bz$$

are the roots of the equation

$$4t^3-3(a^2+b^2+c^2)(x^2+y^2+z^2)\,t-54abc\,xyz$$
$$-2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)=0,$$

when $a+b+c$ and $x+y+z$ vanish.

Solution by K. B. Madhava.

The expression on the left hand side of the identity is symmetrical separately in a, b and x, y and between themselves and is homogeneous in them of the third degree.

We see that $a-b$ is a factor, because the first member becomes identically zero when in it we put $a=b$.

Again, since we can write the expression in the form

$$2\,\{\,(2a+b)x+(a+2b)y\,\}^3-6\,\{\,(2a+b)x+(a+2b)y\,\}\times$$
$$\{\,\tfrac{1}{2}\,(2a+b)(a+2b)-\tfrac{3}{2}ab\,\}\,(x^2+y^2+z^2)-27ab\,xy\,\,(2a+b-a)(x+y)$$

we notice that $2a+b$ is a factor.

Similarly $a+2b$ is a factor.

From this we can also infer that $x-y$, $2x+y$, $2y+x$ are also factors. We can also satisfy ourselves that the constant multiplier is unity.

Hence the identity.

To deduce the second result, we have $a+b+c=0$ and $x+y+z=0$, $(a+b)(x+y)=cz$; $2a+b=a-c$; $2b+a=b-c$; $a^2+ab+c^2=\frac{1}{2}(a^2+b^2+c^2)$,&c.

Inserting these, we have

$$4(ax+by+cz)^3-3(a^2+b^2+c^2)\,(x^2+y^2+z^2)\,\,(ax+by+cz)-$$
$$54\,abc\,xyz-\ldots=0$$

which shows that $ax+by+cz$, and on account of symmetry $bx+cy+az$, $cx+ay+bz$ also, are the three roots of the given cubic in t.

Question 659.

(M. Bhimasena Rao) :—Show that

$$\text{(i)}\ (1+e^{-\pi})(1+e^{-2\pi})(1+e^{-3\pi})\ldots=e^{\frac{\pi}{24}}\left(\frac{1}{2}\right)^{\frac{1}{8}}.$$

$$\text{(ii}\ (1+e^{-\pi\sqrt{2}})(1+e^{-2\pi\sqrt{2}})(1+e^{3\pi\sqrt{2}})\ldots=e^{\frac{\pi\sqrt{2}}{24}}\left(\frac{1}{2}\right)^{\frac{1}{4}}.$$

Solution by K. B. Madhava.

Without using Elliptic Functions these results may be got from the simpler theory of Infinite Products as in Bromwich, Chap. VI, Ex. (14—20).

Borrowing his notation

$$q_1=\Pi(1+q^{2n})\;;\; q_2=\Pi(1+q^{2n-1})\;;\; q_3=\Pi(1-q^{2n-1})$$

we have $q_1q_2q_3=1$ (Ex. 14), $q_2^8=q_3^8+16qq_1^8$ (Ex. 20).

Now
$$q_2=e^{-\frac{\pi}{24}}2^{\frac{1}{4}} \text{ (Q. 608)}$$

$$q_2^8=q_1^{-8}q_2^{-8}+16q\,q_1^8$$

$$4e^{-\frac{1}{3}\pi}=\tfrac{1}{4}e^{\frac{1}{3}\pi}q_1-8+16q_1^8e^{-\pi}$$

$$(8e^{-\frac{2}{3}\pi}q_1^8-1)^2=0, \text{ i.e. } q_1=e\,e^{\frac{1}{12}\pi}2^{-\frac{3}{8}}\;;\; q_1q_2=e^{\frac{1}{24}\pi}2^{-\frac{1}{8}}$$

which is result (i).

Again $q_2=e^{-\frac{\pi\sqrt{2}}{24}}.2^{\frac{1}{8}}(1+\sqrt{2})^{\frac{1}{8}}$

and the above equation for q_1^8 solved will give q_1q_2 in this case

$$=e^{\frac{\pi\sqrt{2}}{24}}\left(\frac{1}{2}\right)^{\frac{1}{4}}$$

Question 663.

(S. Krishnaswami Aiyangar, B. A.) :—Establish the following relation with respect to Bernoulli's numbers :

$$\frac{n}{2}=c_2B_1-c_4B_3\cdot 4+c_6B_5\cdot 4^2\ldots(-)^{n-1}c_{2n}B_{2n-1}4^{n-1},$$

where $c_r={}_{2n+1}C_r$.

Solution by K. B. Madhava and J. C. Swaminarayan, M.A.

We have

$$1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\ldots\ldots(-)^n\frac{\theta^{2n}}{(2n)!}\ldots=\cos\theta=\sin\theta\times\cot\theta$$

$$=\left[\theta-\frac{\theta^3}{3!}\ldots(-)^n\frac{\theta^{2n+1}}{(2n+1)!}\ldots\right]\left[\frac{1}{\theta}-\frac{2^2B_1}{2!}\theta-\frac{2^4B_3}{4!}\theta^3\ldots-\frac{2^{2n}B^{2n-1}}{(2n)!}\theta_{2n-1}\ldots\right]$$

Equating the coefficient of θ^{2n}, we have

$$\frac{1}{(2n)!}-\frac{1}{(2n+1)!}=\frac{2^2B_1}{2!\,(2n-1)!}-\frac{2^4B_3}{(4!\,2n-3)!}\ldots(-)^{n-1}\frac{2^{2n}B_{2n-1}}{2n!}$$

i.e. $$2n=2^2.c_2B_1-2^4c_4B_3+\ldots(-)^{n-1}2^{2n}c_{2n}B_{2n-1}$$

i.e. $$\frac{n}{2}=c_2B_1-c_4B_3\cdot 4+c_6B_5\cdot 4^2\ldots(-)^{n-1}c_{2n}B_{2n-1}4^{n-1}.$$

Question 664.

(S. Krishnaswami Aiyangar, B.A.) :—Prove that

$$\sum_{1}^{\infty}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^{2}\cdot\frac{1}{n}=\frac{16}{\pi}-4.$$

Solution by J. C. Swaminarayan, M.A.

We have to prove that

$$\sum_{1}\left\{\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\right\}^{2}\cdot\frac{1}{n}=\sum_{1}^{\infty}\int_{0}^{1}\int_{0}^{1}\frac{x^{n}y^{n}\,dx\,dy}{n\sqrt{(1-x)(1-y)}}=16-4\pi.$$

Now
$$\sum_{1}^{\infty}\int_{0}^{1}\int_{0}^{1}\frac{x^{n}y^{n}\,dx\,dy}{n\sqrt{(1-x)(1-y)}}\quad\int_{0}^{1}\int_{0}^{1}\frac{-\log(1-xy)}{\sqrt{(1-x)(1-y)}}\,dx\,dy$$

$$=\int_{0}^{1}\frac{dx}{\sqrt{1-x}}\left[\Big\{2\sqrt{1-y}\log(1-xy)\Big\}_{0}^{1}+2\int_{0}^{1}\frac{x\sqrt{1-y}}{1-xy}dy\right]$$

$$=\int_{0}^{1}\frac{2dx}{\sqrt{1-x}}\left[2\int_{0}^{1}\left\{1-\frac{1-x}{1-x+xz^{2}}\right\}dz\right],\ \text{where } 1-y=z^{2}$$

$$=4\int_{0}^{1}\left[1-\sqrt{\frac{1-x}{x}}\tan^{-1}\left(\sqrt{\frac{x}{1-x}}\right)\right]\frac{dx}{\sqrt{1-x}}$$

$$=4\int_{0}^{1}\frac{dx}{\sqrt{1-x}}-4\int_{0}^{1}\frac{\tan^{-1}\sqrt{\frac{x}{1-x}}}{\sqrt{x}}dx$$

$$=\left[-8\sqrt{1-x}\right]_{0}^{1}-4\int_{0}^{\frac{\pi}{2}}\frac{\tan^{-1}(\tan\theta)}{\sin\theta}\times 2\sin\theta\cos\theta\,d\theta$$

[where $x=\sin^{2}\theta$]

$$=8-8\int_{0}^{\frac{\pi}{2}}\theta\cos\theta\,d\theta$$

$$=8-8\Big[\theta\sin\theta+\cos\theta\Big]_{0}^{\frac{\pi}{2}}$$

$$=8-8\left[\frac{\pi}{2}-1\right]$$

$$=8+8-4\pi=16-4\pi.$$

Question 672.

(S. KRISHNASWAMI AIYANGAR, B.A.) :—The vertex A of the triangle of reference lies without, on, or within the director circle of the maximum inscribed ellipse according as the angle A is acute, right or obtuse.

Remarks and Solution (1) *by K. B. Madhava*, (2) *by Narasinga Rao, B.A.*

(1) That if the vertex A lay on the director circle of the maximum inscribed ellipse (in fact of any inscribed conic) A should be a right angle is obvions and needs no proof.

Also from the elementary properties of the figure it is clear that the squares on the tangents from the vertices to the director circle are

$$\tfrac{1}{3}\, bc \cos A\ ;\ \tfrac{1}{3}\, ca \cos B\ ;\ \tfrac{1}{3}\, ab \cos C$$

so that the equation of the director circle (in areals) is

$$a^2yz+b^2zx+c^2xy=\tfrac{1}{3}(x+y+z)(bc \cos A\, x+ca \cos B y+ab \cos C z).$$

Thus the power of the vertex A is $\frac{1}{3}\, bc \cos A$, which is positive, zero, or negative according as A is acute, right or obtuse. Hence the theorem.

The theorem is indeed true for any inscribed conic as is clear from the fact (See Asquith, P. 421, Ex. 33) that if in areal tangentials the equation of any inscribed conic be $fmn+gnl+hlm=0$, the equation of the director circle is

$$2(f+g+h)(a^2yz+b^2zx+c^2xy)=$$
$$(x+y+z)\left\{ (b^2+c^2-a^2)fx+(c^2+a^2-b^2)gy+(a^2+b^2-c^2)hz \right\}.$$

(2) Let α be the angle between the tangents drawn from any point (x, y) to the ellipse $S=0$.

Then α is a function of x and y, say $f(x, y)$.

Consider now the loci $\alpha=$ constant.

We have the following results respecting these :—

(i) These loci are concentric and coaxial ovals.

(ii) If P be a point on a fixed line through the centre O and distant r from it, then $\alpha=\phi(r)$ is a continually increasing or continually decreasing function of r, for, a stationary value of $\phi(r)$ corresponds to OP being a tangent at P to some curve of the system which is impossible. Also $\phi(r)$ vanishes for large values of r.

Hence $\phi(r)$ is a decreasing function of r.

(iii) All these loci are closed curves and no two can intersect in a real point; for, if they do, the " α " of the common point will have two different values. Hence they lie one within the other. For values of α very near π the locus approximates to $S=0$ while for very small values of α, it tends to the line at infinity.

(iv) If P lies between the curves $\alpha=\alpha_1$ and $\alpha=\alpha_2$ then Pα lies between α_1 and α_2.

Let OP cut the two loci S_1 and S_2 in P_1 and P_2. Then P lies between P_1 and P_2 and therefore OP lies between OP_1 and OP_2. But $\phi(r)$ is monotonic.

$$\therefore \qquad \phi(r_2) < \phi(OP) < \phi(r_1)$$

$$i.e. \qquad \alpha_2 < \alpha < \alpha_1.$$

The converse proposition that if $\alpha < \alpha_1$, P lies without S_1, is proved on the same lines.

The given question is a particular case of this, for when $\alpha=\frac{1}{2}\pi$, Sα is the director circle. Hence if a Δ circumscribes an Ellipse, any vertex will lie within, on, or without the director $\odot$ according as it is obtuse, right, or acute. This is true of a Δ and any inscribed Ellipse and in particular, of its maximum inscribed ellipse.

Question 675.

(K. Appukuttan Erady, M.A.) :—Shew that

$$\text{(i)} \int_0^{\pi} \frac{e^{b\cos\theta}\cos(b\sin\theta)}{1-2a\cos\theta+a^2}d\theta=\frac{\pi e^{ab}}{1-a^2};$$

$$\text{(ii)} \int_0^{\pi} \frac{\log(1-2b\cos\theta+b^2)}{1-2a\cos\theta+a^2}d\theta=\frac{2\pi}{1-a^2}\log(1-ab).$$

(1) *Solution by J. M. Bose, M.A, B.Sc.*

$$\text{(i)} \int_0^{\pi} \frac{e^{b\cos\theta}\cos(b\sin\theta)}{(1-2a\cos\theta+a^2)}\,d\theta.$$

Let

$$I=\int_0^{\pi} \frac{e^{b\cos\theta}\cos(b\sin\theta)}{1-2a\cos\theta+a^2}\,d\theta$$

$$J=\int_0^{\pi} \frac{e^{b\cos\theta}\sin(b\sin\theta)}{1-2a\cos\theta+a^2}\,d\theta\,;$$

so that

$$I+iJ=\int_0^{\pi} \frac{e^{be^{i\theta}}}{1-a(e^{i\theta}+e^{-i\theta})+a^2}\,d\theta$$

$$=\int_C \frac{ie^{bz}dz}{az^2-(1+a^2)z+a}=i\int_C \frac{e^{bz}dz}{(z-a)(az-1)}, \text{ putting } z=e^{i\theta}$$

where the integral is taken round a circle of unit radius in the z-plane.

The poles of the function are a and $\frac{1}{a}$.

(1) Let $a<1$, in which case the only pole within the contour is $z=a$, and the residue at this pole is $\frac{e^{ab}}{a^2-1}$.

$$\therefore \quad I + i\,J = \tfrac{1}{2}\left\{ 2\pi i \times \frac{ie^{ab}}{a^2-1} \right\} = \frac{\pi e^{ab}}{1-a^2}.$$

Similarly, (2) if $a>1$, then the pole inside the contour is $1/a$, and

$$I + i\,J = \frac{\pi e^{ba}}{a^2-1},$$

so that

$$I = \frac{\pi e^{ab}}{1-a^2}, \text{ and } J=0.$$

(ii) $\displaystyle\int_0^{\pi} \frac{\log(1-2b\cos\theta+b^2)}{(1-2a\cos\theta+a^2)}\,d\theta.$

If we differentiate with respect to b

$$\frac{\partial I}{\partial b} = \int_0^{\pi} \frac{2(b-\cos\theta)d\theta}{(1-2b\cos b+b^2)(1-2a\cos\theta+a^2)}$$

$$= \frac{1}{(1-ab)(b-a)} \int_0^{\pi} \left[\frac{a^2-2ab+1}{1-2a\cos\theta+a^2} - \frac{1-b^2}{1-2b\cos\theta+b^2} \right] d\theta.$$

Integrating each term separately we can get the required result, but the following method is more interesting.

$2\cos\theta = e^{i\theta} + e^{-i\theta}$ and $z=e^{i\theta}$ so that

$$\frac{\partial I}{\partial b} = i\int_0^{\pi} \frac{z^2-2bz+1}{(bz-1)(z-b)(az-1)(z-a)}\,dz,$$

where the integral is to be taken round a circle of unit radius in the z-plane.

If $a<1$, $b<1$, then the two poles within the contour are a and b. The residues at these poles are

$$\frac{a^2-2ab+1}{(ab-1)(a-b)(a^2-1)} \text{ and } \frac{1-b^2}{(b^2-1)(ab-1)(b-1)} \text{ respectively;}$$

so that
$$\frac{\partial I}{\partial b} = \tfrac{1}{2}.\ 2\pi i \times \text{sum of the residues}$$

$$= \frac{-2a\pi}{(1-ab)(a^2-1)},$$

which gives the required result on integration.

Let $a>1$, $b>1$ then the poles are $1/a$ and $1/b$; and the residues are

$$\frac{ab(1-2ab+a^2)}{(b-a)(1-ab)(1-a^2)},\ \frac{ab(1-b^2)}{(a-b)(1-ab)};$$

so that

$$I=\frac{2\pi}{a^2-1}\log(ab-1).$$

(2) *Solution by A. Narasinga Rao, B.A.*

Poisson's formula states

$$\int_0^\pi \left[\frac{f(a+e^{ix})+f(a+e^{-ix})}{1-2c\cos x+c^2}\right]dx=\frac{2\pi}{1-c^2}f(a+c)$$

(Carr's *Synopsis*, 2702).

(i) Put $a=0$, $f(\theta)=e^{b\theta}$ and $c=a$. Then

$$\int_0^\pi \frac{e^{be^{ix}}+e^{be^{-ix}}}{1-2a\cos x+a^2}dx=\frac{2\pi}{1-a^2}e^{ab}.$$

i.e.
$$\int_0^\pi \frac{e^{b\cos x}\left(e^{ib\sin x}+e^{-ib\sin x}\right)}{1-2a\cos x+a^2}dx=\frac{2\pi e^{ab}}{1-a^2}$$

i.e.
$$\int_0^\pi \frac{e^{b\cos x}\cos(b\sin x)}{1-2a\cos x+a^2}dx=\frac{\pi e^{ab}}{1-a^2}.$$

(ii) Put $a=0$, $f(\theta)=\log(1-b\theta)$ and $c=a$.

$$\int_0^\pi \frac{\log(1-be^{ix})+\log(1-be^{-ix})}{1-2a\cos x+a^2}dx=\frac{2\pi}{(1-a^2)}\log(1-ab).$$

i.e.
$$\int_0^\pi \frac{\log(1-2b\cos x+b^2)}{1-2a\cos x+a^2}dx=\frac{2\pi}{1-a^2}\log(1-ab).$$

(3) *Solution by D. Krishna Murti.*

Let the integral (i) be denoted by w.
We can shew that

$$(1-2a\cos\theta+a^2)^{-1}=\frac{1}{1-a^2}+\sum_{r=1}^{\infty}\frac{2a^r\cos r\theta}{1-a^2}\quad \ldots \quad \ldots \quad \text{A}$$

and that

$$e^{b\cos\theta}\cos(b\sin\theta)=\sum_{n=0}^{\infty}\frac{b^n\cos n\theta}{n!}\quad \ldots \quad \ldots \quad \text{B}$$

Consider the integral

$$w_r = \int_0^{\pi} e^{b\cos\Theta} \cos(b\sin\Theta)\cos r\Theta\, d\Theta.$$

With the aid of B, we get $w_r = \dfrac{\pi b^r}{2.r!}$

Now

$$w = \frac{1}{1-a^2}\int_0^{\pi} e^{b\cos\Theta}\cos(b\sin\Theta)\, d\Theta + \sum_{r=1}^{\infty} \frac{2a^r}{1-a^2}.\, w_r \text{ (from A)}$$

i.e.
$$w = \frac{\pi}{1-a^2} + \frac{\pi}{1-a^2}\sum_{r=1}^{\infty} \frac{a^r b^r}{r!}$$

$$= \frac{\pi e^{ab}}{1-a^2}.$$

(ii)
$$\int_0^{\pi} \frac{\log(1-2b\cos\Theta+b^2)}{1-2a\cos\Theta+a^2}.\, d\Theta = \frac{2\pi}{1-a^2}\log(1-ab).$$

We have

$$I = \int_0^{\pi} \frac{\cos r\Theta . d\Theta}{1-2a\cos\Theta+a^2} = \frac{\pi a^r}{1-a^2}, \text{ } r \text{ being an integer.}$$

For,

$$I = \frac{1}{1-a^2}\int_0^{\pi}\cos r\Theta\, d\Theta + \sum_{k=1}^{\infty}\int_0^{\pi} \frac{2a^k\cos k\Theta\cos r\Theta . d\Theta}{1-a^2}$$

$$= \frac{2a^r}{1-a^2}\int_0^{\pi}\cos^2 r\Theta . d\Theta = \frac{\pi a^r}{1-a^2}.$$

Now let u denote the given integral. Then

$$\log(1-2b\cos\Theta+b^2) = -2\sum_{n=1}^{\infty}\frac{b^n\cos n\Theta}{n}.$$

$$\therefore \qquad u = -2\sum_{n=1}^{\infty}\int_0^{\pi}\frac{b^n\cos n\Theta}{n(1-2a\cos\Theta+a^2)}.\, d\Theta$$

$$= -2\sum_{n=1}^{\infty}\frac{\pi a^n b^n}{n(1-a^2)}$$

$$= \frac{2\pi}{(1-a^2)}\log(1-ab).$$

Question 689.

(J. C. Swaminarayan, M.A.) :—If $S_m = 1^m + 2^m + 3^m + \ldots n^m$, prove that

$$7S_6 + 5\,S_4 = 12\,S_1^2 S_2.$$

Solution (1) *by K. B. Madhava, R. Srinivasan, M.A. and S. V. Venkataraya Sastry, M.A., L.T.*; (2) *by R. D. Karve.*

(1) We know

$$S_m = \frac{n^{m+1}}{m+1} + \frac{1}{2}n^m + B_1\frac{m}{2!}.n^{m-1} - B_3\frac{m(m-1)(m-2)}{4!}.n^{m-3} + \ldots$$

where B_1, B_3... are Bernoulli's numbers. (Hall and Knight's *Higher Algebra*, § 406.)

Hence

$$S_1 = \tfrac{1}{2}n\,(n+1),$$

$$S_2 = \tfrac{1}{6}n(n+1)(2n+1),$$

$$S_4 = \frac{n^5}{5} + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n,$$

$$S_6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$$

Making these substitutions we get the required result.

(2) We have

$$(2n+1)^7 - (2n-1)^7 = 896n^6 + 1120n^4 + 168n^2 + 2$$

$$(2n-1)^7 - (2n-3)^7 = 896(n-1)^6 + 1120(n-1)^4 + 168(n-1)^2) + 2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$3^7 - 1^7 = 896.1^6 + 1120.1^4 + 168.1^2 + 2$$

$$\therefore \quad (2n+1)^7 - 1 = 896.S_6 + 1120.S_4 + 168S_2 + 2n,$$

giving

$$224S_6 + 280S_4 + 42S_2$$
$$= n(2n+1)(16n^5 + 48n^4 + 60n^3 + 40n^2 + 15n + 3).$$

Similarly,

$$120S_4 + 60S_2 = 6n(2n+1)(2n^3 + 4n^2 + 3n + 1).$$

Subtracting, we have

$$32(7S_6 + 5S_4) = n(2n+1)(16n^5 + 48n^4 + 48n^3 + 16n^2 - 3n - 3) + 18S_2,$$
$$= n(n+1)(2n+1)(16n^4 + 32n^3 + 16n^2 - 3) + 18S_2$$
$$= 6S_2 \cdot 16n^2(n+1)^2$$
$$= 6S_2 \cdot 16 \cdot 4S_1^2.$$

Thus

$$7S_6 + 5S_4 = 12S_2 \cdot S_1^2.$$

Question 690.

(J. C. Swaminarayan, M.A.) :—If S_r denotes the sum of the reciprocals of the first r odd natural numbers, prove the following relations

(i) $\frac{S_1}{3.4}-\frac{S_1}{3\cdot4}+\frac{S_2}{5\cdot6}-\frac{S_4}{7\cdot8}+\ldots\ldots$

$$=-\frac{\pi^2}{32}+\frac{\pi}{8}\log 2+\frac{1}{2}\left\{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right\}$$

(ii) $\frac{1}{1\cdot2}-\frac{S_1}{3\cdot4}+\frac{S_2}{5\cdot6}-\frac{S_3}{7\cdot8}+\ldots\ldots$

$$=-\frac{\pi^2}{32}+\frac{\pi}{4}+\left(\frac{\pi}{8}-\frac{1}{2}\right)\log 2+\frac{1}{2}\left\{\frac{1}{3^3}-\frac{1}{5^2}+\frac{1}{2^2}-\cdots\right\}$$

Solution by Martyn M. Thomas, K. B. Madhava and K. R. Rama Iyer.

Now $\frac{\tan^{-1}x}{1+x^2}=\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots\right)\left(1-x^2+x^4-x^6+\ldots\ldots\right)$

$$=x-x^3\left(1+\frac{1}{3}\right)+x^5\left(1+\frac{1}{3}+\frac{1}{5}\right)-x^7\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)+$$

$$=x\,S_1-x^3\,S_2+x^5\,S_3-x^7\,S_4+x^9\,S_5-\ldots \qquad \ldots \quad (1)$$

$$\therefore \qquad \frac{1}{2}\frac{d}{dx}\{(\tan^{-1}x)^2\}=xS_1-x^3S_2+x^5S_3-x^7S_4+\ldots\ldots$$

Integrating the relation, we have

$$\tfrac{1}{2}\{\tan^{-1}x\}^2=\frac{x^2}{2}S_1-\frac{x^4}{4}S_2+\frac{x^6}{6}S_3-\frac{x^8}{8}S_4+\ldots$$

Dividing by x^2 and integrating between the limits 0 and 1,

$$\frac{1}{2}\int_0^1\frac{(\tan^{-1}x)^2}{x^2}dx=\left[\frac{x}{1\cdot2}S_1-\frac{x^3}{3\cdot4}S_2+\frac{x^5}{5\cdot6}S_3-\frac{x^7}{7\cdot8}S_4+\ldots\right]_0^1$$

$$=\frac{S_1}{1\cdot2}-\frac{S_2}{3\cdot4}+\frac{S_3}{5\cdot6}-\frac{S_4}{7\cdot8}+\ldots$$

Hence the given series (1)

$$=\tfrac{1}{2}\int_0^1\frac{(\tan^{-1}x)^2}{x^2}dx$$

$$=\tfrac{1}{2}\int_0^{\frac{\pi}{4}}\theta^2\operatorname{cosec}^2\theta\,d\theta$$

$$=\tfrac{1}{2}[-\theta^2\cot\theta]+\int_0^{\frac{\pi}{4}}\theta\cot\theta\,d\theta$$

$$=-\frac{\pi^2}{32}+\Big[\log\sin\theta\Big]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\log\sin\theta\,d\theta$$

$$=-\frac{\pi^2}{32}+\frac{\pi}{4}\log\frac{1}{\sqrt{2}}-\int_0^{\frac{\pi}{4}}\Big[-\log 2-$$

$$\{\cos 2\theta+\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta+\dots\}\Big]d\theta$$

$$=-\frac{\pi^2}{32}-\frac{\pi}{8}\log 2+\frac{\pi}{4}\log 2+\Big[\frac{\sin 2\theta}{2}+\tfrac{1}{2}.\frac{\sin 4\theta}{4}+\tfrac{1}{3}.\frac{\sin 6\theta}{6}+\dots\Big]_0^{\frac{\pi}{4}}$$

$$=-\frac{\pi^2}{32}+\frac{\pi}{8}\log 2+\tfrac{1}{2}\Big[1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\Big]$$

Again multiply (1) throughout by x^2 and x and integrate between the limits 0 and 1 and take the difference. Then

$$\int_0^1\frac{x^2\tan^{-1}x}{1+x^2}\,dx-\int_0^1\frac{x\tan^{-1}x}{1+x^2}dx$$

$$\equiv\Big[\frac{S_1}{4}-\frac{S_2}{6}+\frac{S_3}{8}-\frac{S_4}{10}+\dots\Big]-\Big[\frac{S_1}{3}-\frac{S_2}{5}+\frac{S_3}{7}-\frac{S_4}{9}+\dots\Big]$$

$$=-\frac{S_1}{3.4}+\frac{S_2}{5.6}-\frac{S_3}{7.8}+\dots\dots$$

But

$$\int_0^1\frac{x^2\tan^{-1}x}{1+x^2}\,dx=\int_0^{\frac{\pi}{4}}\theta\tan^2\theta\,d\theta$$

$$=\int_0^{\frac{1}{4}\pi}\theta\sec^2\theta d\theta-\tfrac{1}{2}.\frac{\pi^2}{16}$$

$$=\Big[\theta\tan\theta\Big]-\int_0^{\frac{1}{4}\pi}\tan\theta\,d\theta-\frac{\pi^2}{32}$$

$$=\frac{\pi}{4}+\log\cos\frac{\pi}{4}-\frac{\pi^2}{32}=\frac{\pi}{4}-\tfrac{1}{2}\log 2-\frac{\pi^2}{32}$$

and

$$\int_0^1\frac{x\tan^{-1}x}{1+x^2}\,dx=\int_0^{\frac{\pi}{4}}\theta\tan\theta=\Big[-\theta\log\cos\theta\Big]_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\log\cos\theta\,d\theta$$

$$=-\frac{\pi}{4}\log\frac{1}{\sqrt{2}}+\int_0^{\frac{\pi}{4}}\Big[-\log 2+$$

$$\{\cos 2\theta-\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta-\dots\}\Big]d\theta$$

$$=\frac{\pi}{8}\log 2-\frac{\pi}{4}\log 2+\tfrac{1}{2}\Big\{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\Big\}$$

Hence the series (ii)

$$=\frac{\pi}{4}-\frac{\pi^2}{32}+\Big(\frac{\pi}{8}-\tfrac{1}{2}\Big)\log 2+\tfrac{1}{2}\Big\{\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}-\dots\Big\}.$$

QUESTIONS FOR SOLUTION.

741. (MARTYN M. THOMAS) :—Two particles A and B describe coplanar closed orbits in such a manner that, at every instant, the line joining them represents, in magnitude and direction, the velocity of A in its orbit. Show that the area of the hodograph of A's orbit

$$=(1+n)\left\{ B+\frac{A}{n}-G\left(1+\frac{1}{n}\right) \right\}k^2,$$

where A,B,G represent the areas of the orbits described by the particles A, B and their centre of gravity, n is the ratio of the mass of B to that of A, and k a constant.

742. (MARTYN M. THOMAS) :—Show that a curve and all its pedals, positive and negative, have the same potential, at the pedal origin.

743. (N. SALVA) :—Prove the following construction for the Feuerbach-point :

If AI meet the circumcircle in P and P′ be the reflection of P in BC, then the Feuerbach-point F is the reflection of D in IP′, where D is the point of contact of the incircle and BC.

744. (N. SALVA) ;—If ABCD, AB′CD′ be two harmonic ranges such that D′ is the mid-point of CD, prove that $BC^2=BB'.BD'$.

745. (V. ANANTARAMAN) :—How can 49 diamonds, whose values are in A. P., be divided among 7 persons so that each may get the same number of diamonds, the total value of the diamonds got by each being the same ?

746. (LAKSHMISHANKAR N. BHATT).—Prove that, n being an integer not less than 2,

$$\left\{\frac{1}{n-1}+\frac{1}{2}\cdot\frac{1}{3n-1}+\frac{1.3}{2.4}\,\frac{1}{5n-1}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7n-1}+\ldots ad\ inf.\right\}$$

$$\times\left\{\frac{1}{n+1}+\frac{1}{2}\cdot\frac{1}{3n+1}+\frac{1.3}{2\cdot4}\cdot\frac{1}{5n+1}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7n+1}+\ldots ad\ inf.\right\}$$

$$=\frac{2}{(n-1)(n+1)}+\frac{2}{(3n-1)(3n+1)}+\frac{2}{(5n-1)(5n+1)}+\ldots ad\ inf.$$

747. (K. Appukuttan Eradi, M. A.) :—(A, B, C, D, E) and (P, Q, R, S, T) are two sets of points in space ; prove that

$$\begin{vmatrix} AP^2, & AQ^2, & AR^2, & AS^2, & AT^2 \\ BP^2, & BQ^2, & BR^2, & BS^2, & BT^2 \\ CP^2, & CQ^2, & CR^2, & CS^2, & CT^2 \\ DP^2, & DQ^2, & DR^2, & DS^2, & DT^2 \\ EP^2, & EQ^2, & ER^2, & ES^2, & ET^2 \end{vmatrix} = 0,$$

provided a sphere passes through either set.

748. (K. Appukuttan Eradi, M. A.) .—If a, b, c, d be the sides $2s$ the perimeter, and 2S, 2S′ the sums of pairs of opposite angles of a spherical quadrilateral on a sphere of unit radius, show that

$$\cos\frac{a}{2}\cos\frac{b}{2}\,\text{os}\frac{c}{2}\cos\frac{d}{2}\cos^2\frac{1}{2}(S+S')$$
$$+\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}\sin\frac{d}{2}\sin^2\tfrac{1}{2}(S-S')$$
$$=\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-d).$$

749. (S. Krishnaswami Aiyangar) :—If ρ be the radius of curvature of the curve $r^m = a^m \sin m\theta$, ρ_k the radius of curvature of the corresponding point of its k^{th} negative pedal with regard to the pole, show that

$$(1-mk+m)^{m-1}a^{mk}\rho_k{}^{m-1} = (1-mk)^{m-1}\{(m+1)\rho\}^{mk+m-1}.$$

750. (S. Krishnaswami Aiyangar):—If $(c-z)\,x\sin B = (a-x)y\sin C = (b-y)\,z\sin A$, and $A+B+C=180°$, show that

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

Give also a geometrical interpretation of the result.

751. (Selected) :—Prove that

$$\sum_{-p}^{p}\ \sum_{n=-p}^{p}\left\{\frac{2x+m+n}{(x+m)^2(x+n)^2}\right\}\to 0$$

when $p\to\infty$, provided that all terms for which $m=n$ are omitted from the summation. [Tripos : 1895].

752. (R. Srinivasan, M.A.) :—Four rods, AB, BC, CD, DA are smoothly jointed at the extremities so as to form a quadrilateral ABCD in a vertical plane. A is fixed and C is held at a given vertical height above A. If C is now released and motion ensue, discuss the motion immediately before and after the impact which takes place when BC and CD straighten into a line. (The rods are uniform and AB+DA>BC+CD).

753. (S. Ramanujan) :—If

$$\phi(x)=\tfrac{1}{2}\log 2\pi x-x+\int_{1}^{x}\frac{[t]}{t}\,dt,$$

where $[t]$ denotes the greatest integer in t, show that

$$\operatorname*{Lim}_{x\to\infty} x\,\phi(x)=\frac{1}{24}\,;\quad \operatorname*{Lim}_{x\to\infty} x\,\phi(x)=-\frac{1}{12}.$$

754. (S. Ramanujan) :—Show that

$$\frac{e^{x}\Gamma(1+x)}{x^{x}\sqrt{\pi}}=\sqrt[6]{8x^{3}+4x^{2}+x+\mathrm{E}},$$

where E lies between $\frac{1}{100}$ and $\frac{1}{30}$ for all positive values of x.

755. (S. Ramanujan) :—Let p be the perimeter and e the eccentricity of an ellipse whose centre is C, and let CA and CB be a semi-major and a semi-minor axis. From CA cut off CQ equal to CB and also produce AC to P making CP equal to CB. From A draw AN perpendicular to CA (in the direction of CB). From Q draw QM making with QA an angle equal to ϕ (which is to be determined) and meeting AN at M. Join PM and draw PN making with PM an angle equal to half of the angle APM and meeting AN at N. With P as centre and PA as radius describe a circle cutting PN at K and meeting PB produced at L. Then if

$$\frac{arc\ \mathrm{AL}}{arc\ \mathrm{AK}}=\frac{p}{4\mathrm{AN}}$$

trace the changes in ϕ when e varies from 0 to 1. In particular, show that $\phi=30°$, when $e=0$; $\phi\to 30°$, when $e\to 1$; $\phi=35°$, when $e=.99948$ nearly ; ϕ assumes the minimum value of about $29°\ 58\frac{3}{4}'$, when e is about ·999886 ; and ϕ assumes the maximum value of about $30°\ 44\frac{1}{4}'$ when e is about ·9589.

756. (S. Narayana Aiyar, M. A.) :—If

$$\mathrm{A}_n=\frac{(b-c)(b-cx)(b-cx^2)\ldots\ldots(b-cx^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)},$$

$$\mathrm{B}_n=\frac{(c-a)(c-ax)(c-ax^2)\ldots\ldots(c-ax^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)}$$

and

$$\mathrm{C}_n=\frac{(a-b)(a-bx)(a-bx^2)\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)},$$

show that

$$\mathrm{A}_n+\mathrm{A}_{n-1}(\mathrm{B}_1+\mathrm{C}_1)+\mathrm{A}_{n-2}(\mathrm{B}_2+\mathrm{B}_1\mathrm{C}_1+\mathrm{C}_2)$$
$$+\mathrm{A}_{n-3}(\mathrm{B}_3+\mathrm{B}_2\mathrm{C}_1+\mathrm{B}_1\mathrm{C}_2+\mathrm{C}_3)+\ldots\ldots\ldots=0.$$

List of Periodicals Received.

(From 16th January 1916 to 15th March 1916.)

1. Astrophysical Journal, Vol. 42, Nos. 4 & 5 November & December 1915.
2. Bulletin of the American Mathematical Society, Vol. 22, No. 4, January 1916.
3. Bulletin des Sciences Mathematiques, Vol. 39, November 1915.
4. Liouville's Journal, Vol. 1, No. 2
5. Mathematical Gazette, Vol. 8, No. 120, December 1915, (4 Copies).
6. Mathematical Reprints from Ed. Times, Vol. 28, (2 Copies).
7. Mathematics Teacher, Vol. 8, No. 2, December 1915.
8. Monthly Notices of the Royal Astronomical Society, Vol. 75, Nos. 6, 7, 8 and 9, April to October 1915 and Vol. 76, No. 1, November 1915.
9. Philosophical Magazine, Vol 31, Nos. 181 and 182, January and February 1916.
10. Popular Astronomy, Vol. 24, No. 1, January 1916, (3 Copies)
11. Proceedings of the London Mathematical Society, Vol. 15, No. 1, January 1916.
12. Proceedings of the Royal Society of London, Vol. 92, No. 637, January 1916.
13. School Science and Mathematics, Vol. 16, Nos. 1 and 2, January and February 1916, (2 Copies).
14. Transactions of the American Mathematical Society, Vol. 17, No. 1, January 1916.
15. Transactions of the Royal Society of London, Vol. 216, Nos. 540 and 541.
16. The Tohoku Mathematical Journal, Vol. 8, Nos. 3 and 4, December 1915.
17. Rendiconti del Circolo Mathematio di Palermo, Vol. 40, No. 1.
18. American Mathematical Monthly, Vol. 22, No. 10, December 1915.

The Indian Mathematical Society

(Founded in 1907 for the Advancement of Mathematical Study and Research in India.)

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CONTENTS.

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All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

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PROGRESS REPORT.

1. The following Presents to the Library have been received :—

1. *Madras University Calendar for* 1916, Vols. I, II and III.

2. *Historical Introduction to Mathematical Literature*: by G. A. Miller. New York, Macmillan Co., 1916, pp. xiii+302.

2. On page 24 of Prof. Miller's interesting book, there is a reference to our Society and the Journal in the following terms :—

"In view of the fact that Asia took practically no part in the development of mathematics in modern times up to the beginning of the twentieth century, it is interesting to note another thriving mathematical journal which was started with its borders during the period under consideration. This periodical is entitled *The Journal of the Indian Mathematical Society*, and it was started at Madras, India, in February 1909. The fact that it is the official organ of a society founded in 1907 for the advancement of mathematical study and research in India makes the journal the more interesting and increases its opportunities for usefulness."

Again on page 51 mention is made of the **Indian Mathematical Society** among the **Leading National Mathematical Societies** of the world.

POONA,
31st May 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

The Central Point.

By R. Vythynathaswamy.

[In connection with the present note reference may be made to Bell's *Co-ordinate Geometry* or to Frost's *Solid Geometry* or to Salmon's *Analytical Geometry*, where analytical proofs of the properties of the *central point* will be found. The discussion given here, being mainly geometrical, will be found comparatively simpler].

1. A plane through a generator of a scroll touches the scroll at one and only one point on the generator.

In what follows we shall denote the given generator by PQ, the consecutive generator by P′Q′, where PP′, QQ′ are both $\perp$ to PQ. Any plane through PQ cuts P′Q′ in a point which ultimately tends to a definite point in PQ. Hence it is a tangent plane at one and only one point in PQ.

2. The cross-ratio of four points on a generator is equal to the cross-ratio of the tangent planes at these points. (Salmon, § 470).

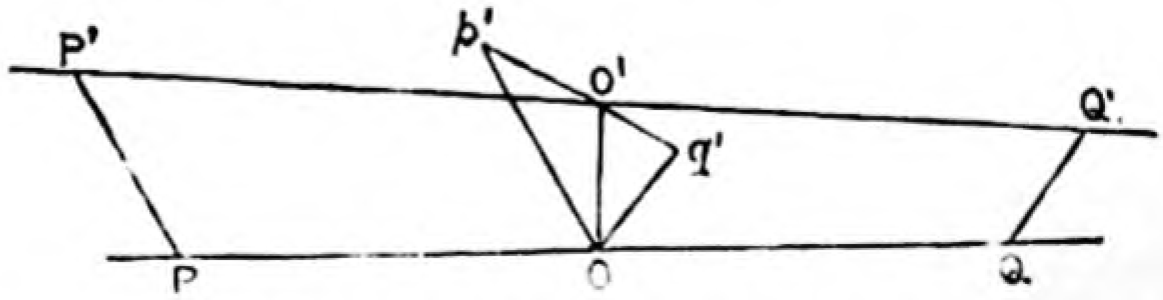
For, by § 1 an (1,1) correspondence exists between tangent planes and the points of contact.

3. The points of contact of perpendicular tangent planes through the same generator are in involution.

For, the pairs of perpendicular tangent planes are themselves in involution.

4. The centre of this involution is *the central point* defined as the point of nearest approach of the given generator and its *consecutive*.

Let OO′ be the shortest distance between PQ and P′Q′. Draw OL parallel to P′Q′. As Q proceeds to ∞ along the line, QQ′ will ultimately lie in the plane POL. Hence QQ′ is ultimately $\perp$ to OO′; in other words, since OQQ′ is in the limit the tangent plane at Q, we see that the tangent planes at the central point and the point at ∞ are perpendicular, which proves the theorem.



5. The constant of this involution is equal to the square of the parameter of distribution.

Let P, Q be points on the given generator, the tangent planes at which are ⊥, let P′Q′ be the consecutive generator where PP′, QQ′ are both ⊥ to PQ. Then PP′, QQ′ are mutually perpendicular. Denoting by $p'q'$ the projection of P′ Q′ on a plane through the shortest distance ⊥ to PQ, it is readily seen that the △ Opq', is right-angled at O and that OO′ is the perpendicular on the hypotenuse. Hence, if $OO'=\delta$ and θ be the angle between the generators,

$$\begin{aligned}\delta^2=OO'^2&=p'O'.O'q'\\&=PO.\ OQ.\ \sin^2\theta.\ \sec^2\theta.\\&=PO\cdot OQ\cdot\theta^2.\end{aligned}$$

$$\therefore\qquad OP\cdot OQ=-\frac{\delta^2}{\theta^2}=-p^2,$$

where p is the parameter of distribution.

Cor. If δ_1, δ_2 be the distances between the generators at corresponding points, then from the right-angled △ $Op'q'$

$$\frac{1}{\delta^2}=\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}.$$

6. The following is another method, substantially the same in principle, of dealing with the central point.

Let x be the distance from a fixed origin of any point in the generator, and $t=\tan\alpha$ be the tangent of the angle made by the normal at this point with a fixed direction ⊥ to the generator. Then since either x or t may be taken as the co-ordinate of a point in the generator, there exists a homographic relation between x and t. Consequently we have the identical relation

$$\begin{vmatrix}1 & x_1 & t_1 & x_1t_1\\ 1 & x_2 & t_2 & x_2t_2\\ 1 & x_3 & t_3 & x_3t_3\\ 1 & x_4 & t_4 & x_4t_4\end{vmatrix}=0.$$

Now choose the fixed direction to be ⊥ to the normal at the point at ∞, the fixed origin to be the point (which we shall call the central point) defined by $t=0$. In the above relation, let x_3, x_4 refer to the origin and the point at ∞ respectively, so that

$$x_3=0,\ t_3=0,\ x_4=\infty,\ t_4=\infty.$$

Substituting, we find

$$\frac{x_1}{t_1}=\frac{x_2}{t_2}=p\ \text{(say)}$$

(1) Putting $t_1 t_2=-1$, we have $x_1 x_2=-p^2$, which shews that the points of contact of $\perp$ tangent planes belong to an involution the centre of which is the central point.

(2) This central point is also the point of the nearest approach of the given generator and its consecutive. The relation

$$\frac{1}{\delta^2}=\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}$$

proved in § 5. *Cor.*, holds for the central point and the point at ∞. Since δ_2 at the point at ∞ is infinite relatively to δ, we must have $\delta_1=\delta$ or the central point is the point of nearest approach.

(3) The quantity p is the *parameter of distribution.*

For a point very near the central point, it is easily seen by projection, as in § 5, that $t_1=\dfrac{x_1 \sin \theta}{\delta}$.

$$\therefore \qquad p=\frac{x_1}{t_1}=\frac{\delta}{\theta}=\text{parameter of distribution.}$$

7. From the homographic relation between t and x, we readily deduce the following :—

(1) If two scrolls have a common generator and intersect at the same angle α at three points in it, their angle of intersection at any other point is also α.

Let t, t' be the tangents of the angles made by the normals to the scrolls at any point in the generator with a fixed direction $\perp$ to the generator. Since t, t' are each homographically related to x, they must be homographically related to each other. Hence if t' is a certain function of t at *three* points it must be the same function of t at all other points, which proves the theorem.

The theorem includes the particular cases of touching and intersecting orthogonally. It is further evident that the central points and parameters of distribution of the common generator regarded as belonging to the two scrolls are the same.

(2) If two scrolls have a common generator there are two points in the generator at which they cut at a given angle α.

Let $t=\tan \beta$, $t'=\tan \beta'$ represent the tangents of the angles made by the normals to the two scrolls at a common point with a fixed direction $\perp$ to the generator. Let $t''=\tan(\beta+\alpha)$. Since there is a homographic relation between t and t' and also between t and t'', there holds a relation of the form $t''=\dfrac{a\,t'+b}{c\,t'+d}$. Putting $t''=t'$ in this, we see that there are two points at which the scrolls cut at an angle α.

In particular, there are two points at which they touch and two points at which they cut orthogonally.

(3) *The normals to a scroll at points on a generator generate a hyperbolic paraboloid.* (Salmon, § 472).

Consider the hyperbolic parabolid determined by the normals *at three points* on the generator. This paraboloid has the given generator in common with the scroll and cuts the scroll orthogonally at these three points. Hence it cuts the scroll orthogonally at any other point K in the generator. Thus the tangent plane to the paraboloid at K is the plane containing the generator and the normal at K. But the other generator through K to the paraboloid must lie in the tangent plane and be ⊥ to the given generator. Hence it must be the normal at K. Thus this paraboloid contains the normals at all points in the generator.

(4) *Consecutive normals to the general surface.*

Let R_1, R_2 be the centres of principal curvature at O to a surface, C X, C Y the axes of the indicatrix; C P, C P′ diameters equally inclined to the axes. It is evident from considerations of symmetry that the normals at P, P′ are equally inclined to the normal at O (this is evident if we consider these normals as the normals to the enveloping cone touching the surface along the indicatrix and having its axis along O R_1). Since further C P′ = CP, it follows that the point of nearest approach (L_1) of normals to the normal at O is the same. Let CQ be the diameter conjugate to CP; P_1, P_2 the centres of curvature of the normal sections OCP, OCQ. Since the curvatures of the sections OCP, OCP′ are the same, it follows that given L_1 we can find a unique position for P_1 and therefore also for P_2. Hence an (1, 1) correspondence exists between L_1 and P_2. By this correspondence R_1 is carried to R_2 and R_2 to R_1. Since the correspondence carries two points into each other it must be an involution.

Further the shortest distance between the normals at P and O is parallel to C Q for the tangent at P is ⊥ to both the normals. If CT is an asymptote to the indicatrix, it follows (since CT is self-conjugate) that OT is the shortest distance between the normals at O and T. Since the centre of curvature of the section OCT is at ∞, it follows that, in the above involution, the point corresponding to O is at ∞. Thus O is the centre of the involution.

Hence $$OL_1 \cdot OP_2 = OR_1 \cdot OR_2$$

∴ $$OL_1 = \frac{OR_1 \cdot OR_2}{OP_2}.$$

Let α be the eccentric angle of P and let $r_2 = CQ$. Let R be a point in OR_1 such that a line through R parallel to CX meets the normal at P. Since the shortest distance of the normals at P and O is parallel to CQ, it follows that

$$L_1 R : L_1 O = \tan P\hat{C}X : \tan Q\hat{C}P$$

$$= \frac{b}{a}\cot \alpha : \frac{ab}{a^2 - r_2^2} \cot \alpha$$

$$= \left(1 - \frac{r_2^2}{a^2}\right).$$

$$\therefore \qquad RL_1 = OL_1 - \frac{OP_2}{OR_1} \cdot \frac{OR_1 \cdot OR_2}{OP_2}$$

$$= OL_1 - OR_2.$$

Thus R is same as R_2. *Hence the normal meets the two lines drawn through either centre of curvature perpendicular to the corresponding section.* (Sturm's Theorem).

(5) Defining an asymptotic curve as a curve on the surface, enveloped by inflexional tangents, we have already shewn that the shortest distance between the normals at two consecutive points of an asymptotic curve is the chord joining them. Or the osculating plane of an asymptotic curve is the tangent plane to the surface at the same point.

The torsion of the asymptotic curve at any point is the square root of the specific curvature of the surface at the same point.

The shortest distance between the normal at T and at O is OT and we know by Sturm's theorem, that lines can be drawn through R_1 and R_2 $\perp$ to OR_1 and to each other, to meet the normal at T. Hence if δ θ, be the shortest distance and the angle between the normals, we have

$$\frac{\delta^2}{\theta^2} = OR_1 . OR_2$$

Hence, torsion $= \dfrac{\theta}{\delta}$ = square root of specific curvature.

(6) We can immediately deduce from this the remarkable property :—

The product of the radii of principal curvature at the central point of a generator of a scroll is equal to the square of the parameter of distribution.

A generator of the scroll is an asymptotic curve and should in this connection be regarded as something like a straight twisted wire,

having a definite osculating plane and binormal at each point. The torsion of this at the point $t=\tan\alpha$ is

$$\frac{dt}{dx}\bigg/\frac{dt}{d\alpha}=\frac{\cos^2\alpha}{p},$$

where p is the parameter of distribution. Hence by the previous theorem the specific curvature at this point is $\frac{\cos^4\alpha}{p^2}$. At the central point $\alpha=0$, and we have $p^2=\rho\rho'$

Corollaries :

(i) Consider the scroll generated by the normals along an asymptotic curve. The asymptotic curve will be a line of striction and the parameter of distribution along any normal will be $\frac{1}{\sqrt{S}}$ where S is the corresponding specific curvature. The tangents to this scroll at the centres of principal curvature will be perpendicular.

(ii) The sum of the square roots of the specific curvature at corresponding points of a generator of a scroll is constant and equal to $\frac{1}{p}$.

For, we have
$$\frac{\cos^2\alpha}{p}+\frac{\sin^2\alpha}{p}=\frac{1}{p}.$$

(iii) If D is the central $\perp$ on a generator of a hyperboloid, axes a, b, c we have

$$D^2p=abc.$$

For $p^2=\rho\rho'$ at the central point. Also the tangent plane at the central point is $\perp$ to the central plane through the generator. Hence by the well-known properties of the conicoid, if α, β be the axes of the parallel central section

$$p^2=\rho\rho'=\frac{\alpha^2\beta^2}{D^2}=\frac{a^2b^2c^2}{D^4}.$$

$$\therefore \qquad D^2p=abc.$$

Japanese Mathematics.

The united labours of Dr. D. E. Smith of New York and Mr. Mikami of Japan, both workers of no mean order in the field of the history of Mathematics, have given to the mathematical world a noteworthy work entitled '*A History of Japanese Mathematics*' (Chicago, 1914). The book is eminently well written and aims at a brief survey of the principal features in the native Japanese mathematics or *wasan*, as it is called. Dr. Smith's scholarly acquaintance with the history of European Mathematics is in evidence throughout the work, a close study of which will amply repay the student.

To us in India, familiar with the traditional methods of dealing with the science of mathematics, the work has a peculiar interest. The Japanese problems are more or less similar to those of *Lilavati* and are inspiring.

The book is printed on excellent paper and is profusely illustrated. It consists of 288 pages divided into fourteen chapters and an index. The reader's interest is well sustained and we make no apology for extracting freely from this unique production. We cannot help regretting, with the authors, the decacy of the *wasan* or native Japanese mathematics, as a result of contact with the West and the modern tendency to broaden the science by bringing it to the level of European mathematics.

The following extracts will speak for themselves :—

(1) "Chin Chiu-shao-(1247) gives a method of approximating the roots of numerical higher equations which he speaks of as the *Ling-lung-kae-fung*, "Harmoniously alternating evolution," a plan in which, by the manipulation of the *sangi*,* he finds the root by what is substantially the method rediscovered by Horner in England in 1819. Another writer of the same period, Yang Hwuy, in his analysis of the *Chiu-chang*, gives the same rule under the name of *Tsang-ching-fang*, "Accumulating involution," but he does not illustrate it by solved problems. We are therefore compelled to admit that Horner's method is a Chinese product of the 13th century, and we shall see that the Japanese adapted it in what we called the third period of their mathematical history.

It is also interesting to know that Chu Chi-chieh in the *Szeyuen-Yu-kien* (1303) gives as an "ancient method" the relation of the binomial co-efficients known in Europe as the "Pascal triangle," and that

* [A calculating device used by the Japanese in which square prisms of 7 mm. thick and 5 cm. long, were used.]

among his names for the various monads (unknowns) is the equivalent for *thing*. This is the same as the Latin *res* and the Italian *cosa*, both of which had undoubtedly come from the East. It is one of the many interesting problems in the history of mathematics to trace the origin of this term.

Chu Chi-chieh writes the equivalent of $a+b+c+x$ as is here shewn, except that we use T for the symbol *tai*, and the modern numerals instead of the *sangi* forms. The square of this expression he writes thus:

$$\begin{array}{ccc} & 1 & \\ 1 & \mathrm{T} & 1 \\ & 1 & \end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 0 & 2 & \\ 1 & 0 & \mathrm{T}^2 & 0 & 1 \\ & 2^2 & 0 & 2 & \\ & & 1 & & \end{array}$$

a method that is quickly learned and easily employed.

The single rule laid down in Chu Chi-*chieh's Suan-hsiao Chi-meng* for the use of *sangi* in the solution of numerical equations contains but little positive information. The method is best understood by actually solving a numerical higher equation, but inasmuch as the manipulation of the *sangi* has already been explained in the preceding chapter, the co-efficients will now be represented by modern numerals. The problem which we shall use is taken from the eighth book of the *Tengen Shinan* of Satō Moshun or Shigeharu, published in 1698, and only the general directions will be given, as was the custom. The reader may compare the work with the common Horner method in which the reasoning involved is more clear.

Let it be required to solve the equation

$$11520-432x-236x^2+4x^3+x^4=0.$$

Arrange the *sangi* on the board to indicate the following:

(r)					
(0)	1	1	5	2	0
(1)		—	4	3	2
(2)		—	2	3	6
(3)					4
(4)					1

Here the top line, marked (r), is reserved for the root, and the lines marked (0), (1), (2), (3), (4) are filled with the *sangi* representing the co-efficient of 0th, 1st, 2nd, 3rd, 4th powers of the unknown quantity.

First, advance the 1st, 2nd, 3rd and 4th degree classes 1, 2, 3, 4 places respectively, thus :

(r)					
(0)	1	1	5	2	0
(1)	–	4	3	2	
(2)	–2	3	6		
(3)		4			
(4)	1				

The root will have two figures and the tens' figure is 1. Multiply this 10 by the 1 of class (4) and add it to class (3), thus making 14 in class (3). Multiply this 14 by the root 10 and add it to —236 of class (2), thus making —96 in class (2). Multiply this —96 by the root 10, and add to —432 of class (1), thus making —1392 in class (1). Multiply this —1392 by root 10 and add to 11520 of class (0), thus making —2400. The result then appears as follows :—

(r)				1	
(0)		–2	4	0	0
(1)	–1	3	9	2	
(2)		–9	6		
(3)	1	4			
(4)	1				

Now repeat the process, multiplying the root 10 into class (4) and adding to class (3), making 24 ; multiply 24 by the root and add to class

(2), making 144; multiply 144 by the root and add to class (1), making 48. The result then appears as follows:

(r)				1	
(0)		−2	4	0	0
(1)			4	8	
(2)	1	4	4		
(3)	2	4			
(4)	1				

Repeat the process, multiplying the root 10 into class (4) and adding to class (3), making 34; multiply 34 by the root and add to class (2) making 484.

Again repeat the process, multiplying the root into class (4) and adding to class (3), making 44.

Now move the *sangi* representing the co-efficients of classes (1). (2) (3), (4) to the right 1, 2, 3, 4 places, respectively, and we have:

(r)				1	
(0)		−2	4	0	0
(1)				4	8
(2)			4	8	4
(3)				4	4
(4)					1

The second figure of the root is 2*. Multiply this into class (4) and add to class (3), making 46. Multiply the same root figure, 2, into this class (3) and add to class (2), making 576. Multiply this 576 by

* It is not stated how either figure is ascertained.

2 and add to class (1), making 1200. Multiply this 1200 by 2 and add to class (0), making 0. The work now appears as follows:

(r)				1	2
(0)					
(1)		1	2		
(2)			5	7	6
(3)				4	6
(4)					1

The root therefore is 12.

It may now be helpful to give a synoptic arrangement of the entire process in order that this Chinese method of the 13th century, practised in Japan in the 17th century, may be compared with Horner's method. The work as described is substantially as follows:

Given

$$x^4+4x^3-236x^2-432x+11520=0.$$

$$\begin{array}{lllll}
1+4 & -236 & -432 & +11520 & \\
10 & 140 & -960 & -13920 & \\
\hline
1\ \ 14 & -\ \ 96 & -1392 & -2400 & \\
10 & 240 & 1440 & & \\
\hline
1\ \ 24 & 144 & 48 & -2400 & \\
10 & 340 & & 2400 & \\
\hline
1\ \ 34 & 484 & 48 & 0 & \\
10 & & 1152 & & \\
\hline
1\ \ 44 & 484 & 1200 & & \\
2 & 92 & & & \\
\hline
1\ \ 46 & 576 & & &
\end{array}$$

Chu Chi-chieh also gives rules for the treatment of negative numbers." [pp. 50—56]

(2) "The name of Seki (1642—1708) has long been associated with the *Yenri*, a form of the calculus that was possibly invented by him. It is with greater certainty that he is known for his *tenzan* method, an algebraic system that improved upon the method of the 'celestial element' inherited from the Chinese; for the *Yendan jutsu*, a scheme by which the treatment of equations and other branches of algebra is

simpler than by the methods inherited from China ; and for his work in determinants that ante-dated what has hitherto been considered the first discovery, namely the investigations of Leibnitz." [pp. 94—95].

(3) "Mention should be made of Seki's work on the mensuration of solids, which appeared in two of his manuscripts. He begins by considering the volume of a ring (*kokan* in Japanese), generated by the revolution of a segment of a circle about a diameter parallel to the chord of the segment. He states that the volume is equal to the product of the cube of the chord and the moment of spherical volume (that is, the volume of a unit sphere).

He likewise finds the volume generated by a lune formed by two arcs, the axis being parallel to the common chord. Such work does not seem very difficult at present, but in Seki's time it was an advance over anything known in Japan. These problems were to Japan what those of Cavalieri were to Europe, making a way for the *Katsujutsu* or the method of multiple integration of a later period. * * *

One of the most marked proofs of Seki's genius is seen in his anticipation of the notion of determinants. The school of Seki offered in succession five diplomas representing various degrees of efficiency. The diploma of the third class was called the *Fukudai-menkyo* and represented eighteen or nineteen subjects. The last of these subjects related to the *fukudai* problems or problems involving determinants, and since it appears in a revision of 1683, its discovery antedates this year. Leibnitz (1646—1716) to whom the Western world generally assigns the first idea of determinants, simply asserted that in order that the equations

$$10+11x+12y=0,\ 20+21x+22y=0,\ 30+31x+32y=0$$

may have the same roots the expression

$$10.21.32-10.22.31-11.20.32+11.22.30+12.20.31-12.21.30$$

must vanish. On the other hand Seki treats of n equations. While Leibnitz's discovery was made in 1693 and was not published until after his death, it is evident that Seki was not only the discoverer but that he had a much broader idea than that of his great German contemporary. To show the essential features of his method we may first suppose that we have two equations of the second degree,

$$ax^2+bx+c=0$$
$$a'x^2+b'x+c'=0.$$

Eliminating x^2 we have

$$(a'b-ab')x+(a'c-ac')=0,$$

and eliminating the absolute term and suppressing the factor x we have

$$(ac'-a'c)x+(bc'-b'c)=0.$$

That is, we have two equations of the second degree and transform them into two equations of the first degree, by what the Japanese called the process of folding (*tatamu*). In the same way we may$_{th}$ transform n equations of the nth degree into n equations of the $(n-1)$ the degree. From the latter equations the *wasanka* (the follower of the *wasan* or native mathematics) proceeded to eliminate the various powers of x. Since it was their custom to write only the co-efficients, including all zero co-efficients, and not to equate to zero, their array of co-efficients formed in itself a determinant, although they did not look upon it as a special function of the co-efficients. On this array Seki now proceeds to perform two operations, the *san* (to cut) and the *chi* (to manage). The *san* consisted in the removal of a constant literal factor in any row or column, exactly as we remove a factor from a determinant to-day. If the array(our determinant)equalled zero, this factor was at once dropped. The *chi* was the same operation with respect to a numerical factor.

Seki also expands this array of co-efficients, practically the determinant that is the eliminant of the equations. In this expansion some of the products are positive and these are called *sei* (kept alive), while others are negative and are called *koku* (put to death), and rules for determining these signs are given. Seki knew that the number of terms in the expansion of a determinant of the nth order was n!, and he also knew the law of interchange of columns and rows. Whatever, therefore, may be our opinion as to Seki's originality in the *yenri*, or even as to his knowledge of that system at all, or as to its value, we are compelled to recognize that to him rather than to Leibnitz is due the first step in the theory which afterwards, chiefly under the influence of Cramer (1750) and Cauchy (1812) was developed into the theory of determinants." [pp. 121—6.]

(4) "Takebe (1664—1739) states that Seki was wont to say that calculations relating to the circle were so difficult that there could be no general method of attack. Indeed he says that Seki was averse to complicated theories, while he himself, took such delight in minute analysis that he finally succeeded in his efforts at the quadrature of the circle. It would thus appear that the *yenri* was not the product of Seki's thought, but rather of Takebe's painstaking labour. * * *

He has

$$\tfrac{1}{4}\,a^2=dh\left[1+\sum_{1}^{\infty}\frac{2^{2n+1}(n\,!)^2}{(2n\times 2)\,!}\left(\frac{h}{d}\right)^n\right],$$

for the square of half an arc a in terms of the height h and diameter d of the arc." [p. 147.]

(5) "Matsunaga's *Hōyen Sankyō* (1739) is composed of five books and is devoted entirely to formulas for the circumference and arcs of a circle, no analysis appearing. His first series is as follows :—

$$\frac{\pi^2}{9}=1+\frac{1^2}{3.4}+\frac{1^2.2^2}{3.4.5.6}+\frac{1^2.2^2.3^2}{3.4.5.6.7.8}+\cdots$$

This is followed by

$$\frac{\pi}{3}=1+\frac{1^2}{4.6}+\frac{1^2.3^2}{4.6.8.10}+\frac{1^2.3^2.5^2}{4.6.8.10.12.14}+\cdots$$

a series which is then employed for the evaluation of π to fifty figures. The result is the following :

$\pi=3.14159$ 26535 89793 23846 26433 83279 50288
41971 69399 5751. [p. 160.]

(6) "Ajima (1739—1798) made a noteworthy change in the *yenri*, in that he took equal divisions of the chord instead of the arc, thus simplifying the work materially. Indeed we may say that in this work Ajima shows the first real approach to a mastery of the idea of the integral calculus that is found in Japan, which approach we may put at about the year 1775. * * *

He gets the length of the arc as follows :

$$\text{arc}=c+\frac{1^2}{2.3}\,\frac{c^3}{d^2}+\frac{1^2.3^2}{2.3.4.5}\,\frac{c^5}{d^4}+\frac{1^2.3^2.5^2}{2.3.4.5.6.7}\,\frac{c^7}{d^6}+\cdots$$

from which other formulas may be derived.

* * * *

Thus we at last find in Ajima's work, the calculus established in the native Japanese mathematics, although possibly with considerable European influence. With him the use of the double series again appears, it having been already employed by Matsunaga and Kurushima, and by him the significance of double integration seems first to have been realized. He lacked the simple symbolism of the West, but he had the spirit of the theory, and although his contemporaries failed to realize his genius in this respect it is now possible to look back upon his work and to evaluate it properly. As a result it is safe to say that Ajima brought mathematics to a higher plane than any other man in the eighteenth century, and that had he lived where he could easily have come into touch with contemporary mathematical thought in other parts of the world he might have made discoveries that would have been of far-reaching importance in the science." [pp. 201—205]

(7) "Wada Nei (1787—1840) also turned his attention to the computation of volumes, simplifying Ajima's work on the two intersecting

cylinders and in general developing a very good working type of the integral calculus so far as it has to do with the question of mensuration.

The question of maxima and minima had already been considered by Seki more than a century before Wada's time, the rule employed being not unlike the present one of equating a differential coefficient to zero, although no explanation was given for the method. * * * He (Wada) not only gave the reason for the rule, but carried the discussion still further, including in his theory, the subject of the maximum and minimum values of infinite series." [pp. 224—5]

(8) "Thus closes the old *wasan*, the native mathematics of Japan. It seems as if a subconscious feeling of the hopelessness of the contest with Western science must have influenced the last half century preceding the opening of Japan. There was no really worthy successor of Wada Nei in all this period, and the feeling that was permeating the political life of Japan, that the day of isolations was passing, seems also to have permeated scientific circles." [p. 253]

(9) "At the close of the eighteenth century Shizuki Tadao (1760—1806) began a work entitled *Rekisho Shinsho* consisting of three parts, each containing two books, the composition of which was completed in 1803. The treatise which was never printed is based upon the works of John Keill (1671—1721) Professor of Astronomy at Oxford. The first part treated of the Copernican system of astronomy and the second and third parts of mechanical theories. The latter part of the work may have had its inspiration from Newton's *Principia*. It was the first Japanese work to treat of mechanics and physics, and it is noteworthy also from the fact that the appendix to the third part contains a nebular hypothesis that is claimed to have been independent of that of Kant and Laplace." [p. 263]

(10) "Europe had several thousand years of mathematics back of her when Newton and Leibnitz worked on the calculus,—years in which every nation knew or might know what its neighbours were doing; years in which the scholars of one country inspired those of another. Japan had had hardly a century of real opportunity in mathematics when Seki entered the field. From the standard of opportunity Japan did remarkable work; from the standpoint of mathematical discovery this work was in every way inferior to that of the West.

When, however, we come to execution it is like picking up a box of the real old red lacquer,—not the kind made for sale in our day. In

execution the work was exquisite in a way wholly unknown in the West. For patience, for the everlasting taking of pains, for ingenuity in untangling minute knots and thousands of them, the problem-solving of the Japanese and the working out of some of the series in the *yenri* have never been equalled.

And what will be the result of the introduction of the new (occidental) mathematics into Japan? It is altogether too early to foresee, just as we cannot foresee the introduction of new art, of new standards of living, of machinery, and of all that goes to make the New Japan. If it shall lead to the application of the peculiar genius of the old school, the genius for taking infinite pains, to large questions in mathematics, then the world may see results that will be epoch-making. If, on the other hand, it shall lead to a contempt for the past, and to a desire to abandon the very thing that makes the *wasan* worthy of study, then we cannot see what the future may have in store." [p. 280.]

SHORT NOTES.

The Period of the Elliptic Functions sn u, cn u, dn u.

1. The period K of the above elliptic functions is known to be equivalent to the series

$$\frac{\pi}{2}\left\{1+\left(\tfrac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\dots\right\} \quad \dots \qquad \dots \quad (1)$$

where k is the modulus. [Dixon. *Elliptic Functions*, p. 79.]

This series is the same as $\frac{1}{2}\pi\ F(\frac{1}{2}, \frac{1}{2}, 1, k^2)$, with the usual notation F for a hypergeometric series.

Also, from Gauss' property of the function F, we have

$$(1+y)^{2\alpha} F\{\alpha, \alpha+\tfrac{1}{2}-\beta, \beta+\tfrac{1}{2}, y^2\} = F\left\{\alpha, \beta, 2\beta, \frac{4y}{(1+y)^2}\right\}$$

[Forsyth: *Diffl. Equations*, p. 214, Ex. 5 (ii)]

so that, putting $\alpha=\beta=\frac{1}{2}$, $y=k$, we infer that

$$(1+k)\ F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2) = F\left\{\tfrac{1}{2}, \tfrac{1}{2}, 1, \frac{4k}{(1+k)^2}\right\}$$

Hence the period K for modulus k is connected with the period P for modulus $p=\dfrac{2\sqrt{k}}{1+k}$ by the following simple relation

$$(1+k)K = P. \quad \dots \qquad \dots \qquad \dots \quad (2)$$

In other words, the series for the period admits of transformation into

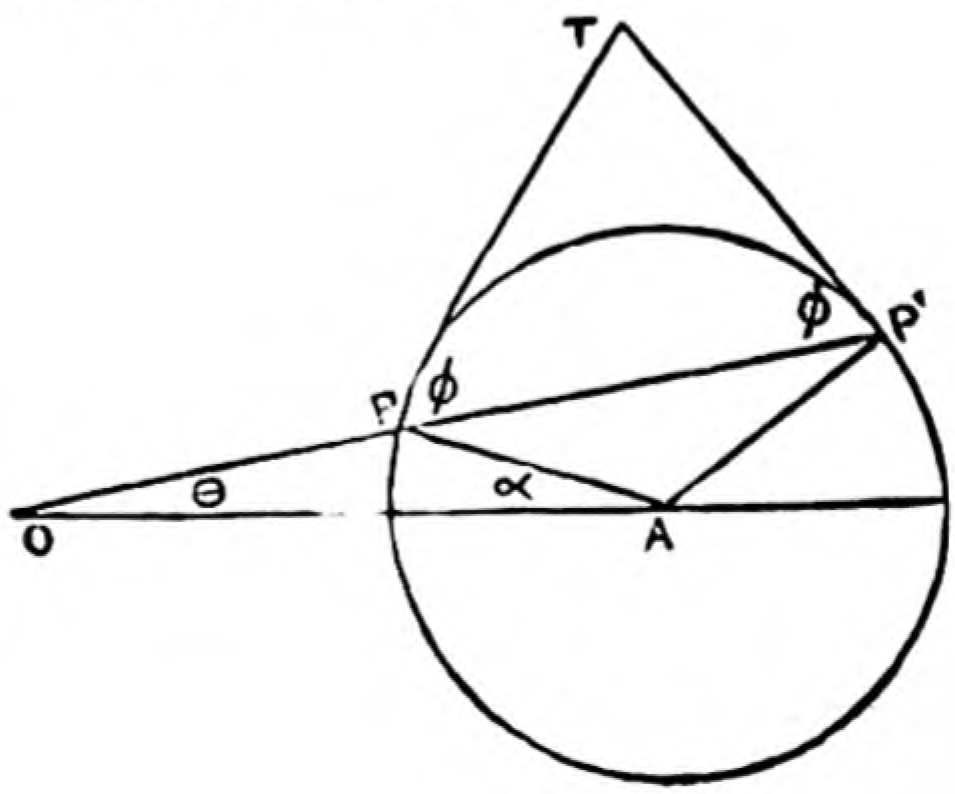
$$\frac{\pi}{4}(1+\sqrt{1-p^2})\left\{1+(\tfrac{1}{2})^2 p^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 p^4+\dots\right\} \quad \dots \qquad \dots \quad (3)$$

2. The following Dynamical proof of the above property is worthy of notice:

Consider the gravitation potential of a thin uniform circular ring (centre A, radius a) at a point O, distant $c>a$ from the centre. With the usual notation, we have

$$V=\mu\nu\left[\int\frac{ds}{r}+\int\left(\frac{ds'}{r'}\right)\right]$$

where ds, ds' are elements at P, P′.



Now

$$\sin\phi=\frac{rd\theta}{ds},\ \sin\phi'=\frac{r'd\theta}{ds'};$$

$\therefore$
$$\frac{ds}{r}=\frac{ds'}{r'}$$

$\therefore$
$$V=2\mu\nu\int\frac{ds}{r}=2\mu\nu\int\frac{d\theta}{\sin\phi}$$

$$=-2\mu\nu\int\frac{ad\phi}{b\cos\theta},\ \text{since } a\cos\phi=c\sin\theta,$$

$$=-2\mu\nu\int\frac{kd\phi}{(1-k^2\cos^2\phi)^{\frac{1}{2}}},$$

where $k=a/c$ and the integration extends over the arc of the ring intercepted between the tangents from O to it.

Hence $$V=4\mu\nu\, k.\mathrm{K}, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$
integrating between proper limits.

Again, from the formula $r^2=a^2+c^2-2ac\cos\alpha$,

$$V=\mu\nu\int_C\frac{ds}{r}=2\mu\nu\int_0^{\pi}\frac{ad\alpha}{(a^2+c^2-2ac\cos\alpha)^{\frac{1}{2}}}$$

$$=4\mu\nu\int_0^{\frac{1}{2}\pi}\frac{ad\lambda}{[(a+c)^2-4ac\cos\lambda^2]^{\frac{1}{2}}},\ [2\lambda=\alpha]$$

$$=4\mu\nu\int_0^{\frac{1}{2}\pi}\frac{ad\lambda}{(a+c)(1-p^2\cos^2\lambda)^{\frac{1}{2}}}$$

$$=4\mu\nu\, k\,\mathrm{P}/(1+k), \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

writing $p=\dfrac{2\sqrt{ac}}{a+c}=\dfrac{2\sqrt{k}}{1+k}$.

Equating (4) and (5), we get
$$\mathrm{P}=(1+k)\,\mathrm{K}$$
which is result (2).

(3) Further, we see that, according to Jacobi's construction, the limiting point L_1 corresponding to the modulus p is the focus of the ellipse of eccentricity k whose auxiliary circle is the fundamental circle of the construction. [Dixon, § 105.]

For

$$p=\frac{2\sqrt{e_1}}{1+e_1},\ \text{if } e_1=\Omega L_1,\ \Omega A$$

$$=\frac{2\sqrt{k}}{1+k}\ \text{by hypothesis.}$$

Hence $e_1=k$, whence the result stated.

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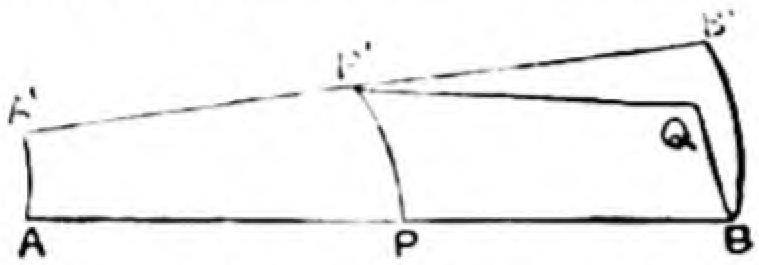
Holditch's Theorem.

1. Various proofs of this elegant theorem are found in Text-books on *Integral Calculus*. [Williamson, § 147 ; Gibson, p. 323 ; Carr, 5244,]

The following demonstration may interest the reader on account of its simplicity :

Let a line P B of length b move to a new position P′B′, where P P′ BB′ are elements of the paths of P and B. Then, resolving the motion of PB into a translation and a rotation, we have



$$\text{area P B B}'\text{P}' = \square\text{PQBP}' + \triangle\text{P}'\text{QB}'$$
$$= ds.b \sin\phi + \tfrac{1}{2} b^2 d\theta, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where ds denotes the element PP′, ϕ the angle A PP′ and $d\theta$ the rotation of BP.

Integrating (1), the area swept over by P B is

$$S = b\int \sin\phi \, ds + \tfrac{1}{2}\int b^2 d\theta. \quad \ldots \quad \ldots \quad (2)$$

If P and B describe complete closed curves of areas (P), (B), one inside the other, obviously

$$S = (B) - (P) \quad \ldots \quad \ldots \quad (3)$$

Hence from (2) and (3)

$$(B) - (P) = b\int_P (ds \sin\phi) + \frac{1}{2} b^2 \int_0^{2\pi} d\theta$$
$$= b \cdot I + \pi b^2, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where I is a quantity depending on the curve (P) and the angle ϕ.

Now, if we take PA $= a$ on BP produced, reasoning as before, we should get

$$(P) - (A) = a.I - \pi a^2. \quad \ldots \quad (5)$$

Eliminating I between (4) and (5), we have

$$a(B) + b(A) - (a+b)(P) = \pi ab(a+b), \quad \ldots \quad (6)$$

which is Holditch's Theorem.

2. In the above demonstration suppose PB, AP are variable and that PB : AP $= b : a$ always. Then P may be regarded as the centre of

masses a, b placed at B, A respectively. Denoting PB, AP by r_2, r_1, we have the formulae

$$(B)-(P)=\int_P r_2 \sin\phi.ds+\tfrac{1}{2}\int_0^{2\pi} r_2^2 d\theta,$$

$$(P)-(A)=\int_P r_1 \sin\phi.ds-\tfrac{1}{2}\int_0^{2\pi} r_1^2 d\theta$$

Since $r_1 : r_2 = a : b$, we deduce

$$a(B)+b(A)-(a+b)(P)=aR_2+bR_1 \quad \ldots \quad \ldots \quad (7)$$

where R_1, R_2 denote the areas described by A and B relative to P.

3. From (7) we get at once the 'Theorem on Areas' mentioned by Mr. R. Vythynathaswamy in a previous issue of this Journal (Vol. VII, p. 96), *viz.*

"If a number of particles $a_1, a_2 \ldots a_n$ of masses $m_1, m_2 \ldots m_n$ describe closed curves of areas $A_1, A_2, \ldots A_n$, then the area G described by their centre of mass is given by

$$G.\Sigma(m_1)=\Sigma(m_1 A_1)-\Sigma(m_1 R_1),$$

where $R_1, R_2 \ldots$ are the areas described by $a_1, a_2 \ldots$ relative to the centre of mass."

4. In this connection the following general property may be noticed:

If a vector AB $(=\rho)$ move according to any law with its ends on given curves $r_1=f_1(\theta_1)$, $r_2=f_2(\theta_2)$; then the element of area described by the vector is approximately

$$\begin{aligned}\delta S &= \text{area AB B}'\text{A}' = \text{OAB}+\text{OBB}'+\text{OB}'\text{A}'+\text{OA}'\text{A}\\ &= \tfrac{1}{2}\{(\rho+\delta\rho)(p+\delta p)-\rho p\}+\tfrac{1}{2}r_2^2\delta\theta_2-\tfrac{1}{2}r_1^2 d\theta_1,\\ &= \tfrac{1}{2}(p\delta\rho+\rho\delta p)+\tfrac{1}{2}r_2^2\delta\theta_2-\tfrac{1}{2}r_1^2\delta\theta_1,\end{aligned}$$

where p denotes the perpendicular from the origin on the vector.

Integrating

$$\begin{aligned}S &= \tfrac{1}{2}\int(p\,d\rho+\rho dp)+\tfrac{1}{2}\int r_2^2 d\theta_2-\tfrac{1}{2}\int r_1^2 d\theta_1\\ &= (B)-(A),\end{aligned}$$

in the case of closed curves.

Hence, in this case,

$$\int(pd\rho+\rho dp)=0.$$

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Note on Question 567.

[The Journal for October 1915 reproduces the clumsy solution of this question found in the text-books. The following discussion of the problem may be of interest to readers of the Journal.]

Let PQ, RS be opposite edges of a uniform tetrahedron of mass M, and let their mid-points be U, V and their lengths $2a$, $2b$; let the length of UV be $2c$, and let G be the mid-point of UV. A plane parallel to PQ and RS, cutting UV in a point whose distance from G towards U is ct, cuts the surface of the tetrahedron in a parallelogram of sides $a(1+t)$, $b(1-t)$, with angles independent of t. Since a prallelogram of mass m is equimomental with particles of masses $m/12$ at the vertices and a particle of mass $2m/3$ at the centre, the tetrahedron is equimomental with a distribution of varying line-density, along the five lines PR, PS, QR, QS, UV, the density in each line being proportional to $1-t^2$, and the total mass of each of the first four lines being $M/12$ and of the fifth line being $2M/3$. Since

$$\int_{-1}^{1} k(1-t^2)dt=4k/3, \int_{-1}^{1} k(1-t^2)t^2\,dt=4k/15,$$

the part of a line of density $k(1-t^2)$ which corresponds to values of t between -1 and 1 has mass n if k is equal to $3n/4$, and the line is equimomental with three particles, one of mass $n/10$ at each end and one of mass $4n/5$ at the mid-point. It follows at once that

'*A uniform tetrahedron of mass M is equimomental with a system of eleven particles, one of mass $M/60$ at each vertex, one of mass $M/15$ at the mid-point of each edge, and one of mass $8M/15$ at the centroid.*'

The deduction of the familiar systems with five particles, of which four are at the vertices, and with seven particles, of which six are at the mid-points of the edges, requires only applications of the theorem that a sytem of three equal particles of mass $m/3$ at the mid-points of the sides of a triangle is equimomental with a system of four particles, one of mass $m/12$ at each vertex and one of mass $3m/4$ at the centroid.

It is evident that the method used here is applicable to many other problems, and it is interesting to use it in the case of a triangle. A uniform line of mass m is equimomental with particles of mass $m/6$ at its end-points and a particle of mass $2m/3$ at its mid-point, and the integrations of t, t^2, and t^3 from 0 to 1 are sufficient to show that a line PQ of mass n whose density is proportional to distance from P has its centroid at the point ot trisection nearer to Q and is equimomental with three particles, one of mass $n/12$ at P, one of mass $n/6$ at Q, and one of mass $3n/4$

at the centroid. It follows that a triangle ABC of mass M is equimomental with a system of seven particles, one of mass $M/12$ at A, two of mass $M/36$ at B and C, one of mass $M/9$ at the mid-point of BC, two of mass $M/8$ at the points of trisection of AB, AC which are further from A, and one of mass $M/2$ at the centroid of the triangle. Superposing three distributions of this form each with total mass $M/3$, we find a symmetrical system composed of thirteen particles, one of mass $5M/108$ at each vertex, one of mass $M/27$ at the mid-point of each side, one of mass $M/24$ at each point of trisection of each side, and one of mass $M/2$ at the centroid, and this system can be replaced immediately by a system of seven particles, one of mass $M/18$ at each vertex, one of mass $M/9$ at the mid-point of each side, and one of mass $M/2$ at the mid-point.

ERIC H. NEVILLE.

The Face of the Sky for July and August 1916.

The Sun

enters Leo on July 22 and Virgo on August 23.

Phases of the Moon.

	July.			August.		
	D.	H.	M.	D.	H.	M.
First Quarter ...	8	5	25 P. M.	7	3	35 A. M.
Full Moon ...	15	10	10 A. M.	13	5	30 P. M.
Last Quarter ...	22	5	3 A. M.	20	6	23 P. M.
New Moon ...	30	7	45 A. M.	28	10	54 P. M.

Eclipses.

There is a lunar eclipse on July 15 and an annular solar eclipse on July 30—both invisible in India.

The Planets.

Mercury which is a morning star in July is in superior conjunction on July 28 and is an evening star in August. It is in conjunction with Venus on July 14, with Saturn on July 22, with the Moon on July 30 and on August 31, with η Gemini on July 12, with η Caneri on July, 28 and with Neptune on July 27.

Venus is in inferior conjunction on July 4 and is stationary on July 25. It attains its greatest brilliance on August 26. It is in conjunction with the Moon on July 27 and on August 24.

Mars is in conjunction with the Moon on July 6 and on August 4.

Jupiter is in quadrature on July 27. It is stationary on August 25. It is in conjunction with the Moon on July 22 and on August 19 at about 5-30 A. M.

Saturn is in conjunction on July 13. It is in conjunction with the Moon on July 29 and on August 25.

Uranus is in opposition on August 10. It is in conjunction with the Moon on July 17 and on August 13.

Neptune is in conjunction on July 25. It in conjunction with the Moon on July 2, on July 29 and on August 26.

V. RAMESAM.

SOLUTIONS.

Question 245.

(H. Br.) Two curves S, S′ are such that if P is any point on S and P′ on S′,

$$PP' = f_1(t) + f_2(t')$$

where t, t' are the variables in terms of which P, P′ may be expressed respectively. Shew that the curves are the focal conics of a system of confocal quadrics.

Solution by N. Durai Rajan.

Let (X, Y, Z) be the co-ordinates of a point P on the 1st curve, expressed in terms of a single variable T. Similarly let x, y, z, t refer to the 2nd curve. Let PQ, pq be the tangents at P, p and let

$\theta = \angle$ between PQ, Pp
$\phi = \angle$ .... pq, pP
$\psi = \angle$ .... pq, PQ
(PQ, pq are not coplanar)

By hypothesis

$$(X-x)^2+(Y-y)^2+(Z-z)^2 \equiv R^2 \equiv [F(T)+f(t)]^2 \quad \dots \quad (1)$$

Differentiating

$$\therefore \quad \Sigma(X-x)\frac{dX}{dT} = R\,\frac{\partial R}{\partial T} = R.\ F'(T) \quad \dots \quad \dots \quad (2)$$

$$\therefore \quad \sum \frac{X-x}{R}\cdot\frac{dX}{dT}\cdot\frac{1}{\sqrt{\left(\frac{dX}{dT}\right)^2+\dots+\dots}} = \frac{F'(T)}{\sqrt{X'^2+Y'^2+Z'^2}}.$$

Hence cos θ is a function of T, and θ is constant if T is constant.

Thus if P be kept fixed and p varies, then the cone described by Pp is right circular, its axis being PQ.

Hence both the curves are curves lying on right circular cones.

Differentiating (2) with regard to T, we get

$$\Sigma(X-x)X''+\Sigma X'^2 = [F'(T)]^2+F''(T)[F(T)+f(t)]$$

$$= [F'(T)]^2+\frac{F''(T)}{F'(T)}\Sigma(X-x)X'$$

$$\therefore \quad \Sigma(x-X)\left(X''-X'.\frac{F''(T)}{F'(T)}\right) = \Sigma X'^2-[F'(T)]^2; \quad \dots \quad (3)$$

i.e. if P be kept fixed, p lies always in a plane.

Hence the curves are plane curves and lie on quadric cones.

In other words, the curves are conics.

We know that the locus of the vertices of right circular cones standing on a given conic is another conic in a perpendicular plane and these conics are the focal conics of a confocal system of quadrics ; whence we infer that the given curves are the focal conics of a set of confocal quadrics.

Verification : The conics are

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\ z=0:\quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1,\ y=0\ ;$$

$$(X,\ Y,\ Z)=\sqrt{a^2-c^2}\cos t,\ \sqrt{b^2-c^2}\sin t,\ 0,$$

$$(x,\ y,\ z)=\sqrt{a^2-b^2}\cosh t',\ 0,\ \sqrt{b^2-c^2}\sinh t''.$$

$$\therefore\ R^2=(\sqrt{a^2-c^2}.\cos t-\sqrt{a^2-b^2}\cosh t')^2+(b^2-c^2)\sin^2 t$$
$$+(b^2-c^2)\sinh^2 t'$$

$$=\cos^2 t\,(a^2-c^2-b^2+c^2)+\cosh^2 t'\,(a^2-b^2+b^2-c^2)$$
$$-2\cosh t'\cos t\sqrt{a^2-b^2},\sqrt{a^2-c^2}+(b^2-c^2)-(b^2-c^2)$$

$$=\left[\sqrt{a^2-b^2}.\cos t-\sqrt{a^2-c^2}.\cosh t'\right]^2.$$

By equating $\dfrac{\partial^2 R}{\partial T\partial t}=0$, we get on reduction the theorem

$$\cos\psi=\cos\theta.\cos\phi.$$

Question 353.

(S. RAMANUJAN) :—If n is any positive odd integer shew that

$$\int_0^\infty \frac{\sin nx}{\cosh x+\cos x}\,\frac{dx}{x}=\frac{\pi}{4}$$

and hence prove that

$$\frac{\pi}{8}=\frac{1}{\left(\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}\right)}-\frac{1}{\left(\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}\right)}+\ldots$$

for all odd values of n.

Solution by A. C. L. Wilkinson, M. A., F. R. A. S.

To evaluate

$$\int\frac{\sin x}{\cosh x+\cos x}\,\frac{dx}{x}.$$

Consider

$$\int \frac{e^z - e^{iz}}{e^z + e^{iz}} \frac{dz}{z}$$

taken over the following contour; (1) the x-axis from O to R, (2) the quadrant of the circle centre the origin and radius R, (3) the y-axis from R to O, where R tends to infinity. The infinities of the function are given by

$$z\,(1-i) = (2n+1)\,i\pi, \text{ or } 2z = -(2n+1)\,\pi\,(1-i);$$

and all these lie on the line $y+x=0$.

Let (α, β) be any point on the circular quadrant, so that

$$\alpha + i\beta = \mathrm{R}e^{i\theta},$$

then

$$\int_0^\infty \frac{e^x - e^{ix}}{e^x + e^{ix}} \frac{dx}{x} + \int_0^{\frac{1}{2}\pi} \frac{e^{\alpha+i\beta} - e^{-\beta+i\alpha}}{e^{\alpha+i\beta} + e^{-\beta+i\alpha}}\, i d\theta - \int_0^\infty \frac{e^{ix} - e^{-x}}{e^{ix} + e^{-x}} \frac{dx}{x} = 0.$$

The function in the second integral is equal to 1 everywhere on the boundary of the infinite quadrant, whence, collecting the first and third integrals together, we have

$$\int_0^\infty \frac{\sin x}{\cosh x + \cos x} \frac{dx}{x} = \frac{\pi}{4}.$$

For the second part it is sufficient to establish that

$$\int_0^\infty \frac{2\cos 2rx \sin x}{\cosh x + \cos x} \frac{dx}{x} = 0$$

for integral values of r.

Consider

$$\int_0^\infty \frac{\sin x\,(1-\cos 2\,rx)}{\cosh x + \cos x} \frac{dx}{x}.$$

The integral is the same as

$$\int_0^\infty \frac{2\,(1-\cos 2rx)}{x} \{ e^{-x}\sin x - e^{-2x}\sin 2x + e^{-3x}\sin 3x - \dots \}\, dx$$

$$= \sum_{n=1}^{n=\infty} \int_0^\infty (-)^{n-1}. \frac{e^{-nx}}{x}. \{ 2\sin nx - \sin (2r+n)x + \sin (2r-n)\,x \}\, dx$$

$$= \sum_{n=1}^{n=\infty} (-)^{n-1} \left\{ \frac{\pi}{2} - \tan^{-1}\frac{2r+n}{n} + \tan^{-1}\frac{2r-n}{n} \right\}$$

$$\text{since} \int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$$

$$= \sum_{n=1}^{n=\infty} (-)^{n-1} \left\{ \frac{\pi}{2} - \tan^{-1}\frac{n^2}{2r^2} \right\}$$

$$= \sum_{n=1}^{n=\infty} (-)^{n-1} \tan^{-1}\frac{2r^2}{n^2} = \frac{\pi}{4}$$ (Bromwich, *Infinite Series*, p. 259, Ex. 34),

whence

$$\int_0^\infty \frac{\sin x \cos 2rx}{\cosh x + \cos x}\frac{dx}{x} = 0, \text{ and } \int_0^\infty \frac{\sin nx}{\cosh x + \cos x}\frac{dx}{x} = \frac{\pi}{4}, \text{ for } n \text{ odd.}$$

Consider

$$\int \frac{1}{\left(\cosh\frac{z}{n} + \cos\frac{z}{n}\right)} \left(\frac{1}{\cosh z} + \frac{1}{\cos z}\right)\frac{dz}{z}$$

where n is an odd integer. This remains finite for the points at which $\cosh\frac{z}{n} + \cos\frac{z}{n}$ vanishes, since when n is odd $\left(\cosh\frac{z}{n} + \cos\frac{z}{n}\right)$ is a factor of $\cosh z + \cos z$.

Thus the infinities of the function we are integrating are

$$z = 0,\ z = \pm\frac{(2r+1)\pi}{2},\ z = \pm\frac{(2r+)i\pi}{2}$$

where r is a positive integer.

Integrate over the following contour

(1) an infinitestinal circle surrounding $z=0$

(2) circles surrounding $z = \pm\frac{(2r+1)\pi}{2}$ and $z = \pm\frac{(2r+1)i\pi}{2}$

(3) a circle of infinitely great radius R so chosen as not to pass indefinitely near the points $\frac{(2r+1)\pi}{2}$, $\frac{(2r+1)i\pi}{2}$, $\frac{(2s+1)\pi}{2}(1+i)$.

The last condition is required in the discussion of the integral over the circle of radius R even though it is not a point at which the inte-

gral becomes infinite. That this is possible is easily seen. For take $R=k\pi$, where k is an integer and suppose $k\pi=\frac{(2s+1)\pi}{2}\sqrt{2}+\epsilon$, then $R=(k+1)\pi$ satisfies our conditions.

The integral round the circle $z=\frac{(2r+1)\pi}{2}$ gives

$$2\pi i\left\{\frac{1}{\cosh\frac{(2r+1)\pi}{2n}+\cosh\frac{(2r+1)\pi}{2n}}\cdot\frac{-1}{\sin\frac{(2r+1)\pi}{2}}\cdot\frac{1}{\frac{(2r+1)\pi}{2}}\right\}$$

$$\frac{4i}{2r+1}\cdot\frac{1}{\cosh\frac{(2r+1)\pi}{2n}+\cos\frac{(2r+1)\pi}{2n}}.$$

The integral round the circle $z=\frac{(2r+1)i\pi}{2}$ gives

$$2\pi i\left\{\frac{1}{\cos\frac{(2r+1)\pi}{2n}+\cosh\frac{(2r+1)\pi}{2n}}\cdot\frac{1}{\sinh\frac{(2r+1)\pi i}{2}}\cdot\frac{1}{\frac{(2r+1)\pi i}{2}}\right\}$$

$$=(-)^{r-1}\cdot\frac{4i}{2r+1}\cdot\frac{1}{\cosh\frac{(2r+1)\pi}{2n}+\cos\frac{(2r+1)\pi}{2n}}.$$

The integral round $z=0$ gives $2\pi i$.

Denoting by S the series

$$\frac{1}{\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}}-\frac{1}{3}\frac{1}{\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}}+\frac{1}{5}\frac{1}{\cosh\frac{5\pi}{2n}+\cos\frac{5\pi}{2n}}-\ldots,$$

we have

$$-16\,i\,S+2\pi i=\int_0^{2\pi}\frac{1}{\cosh\frac{z}{n}+\cos\frac{z}{n}}\left(\frac{1}{\cosh z}+\frac{1}{\cos z}\right)id\theta,$$

where $z=Re^{i\theta}=\alpha+i\beta$.

Now the functions

$$2\cosh z=e^{\alpha+i\beta}+e^{-\alpha-i\beta}$$

$$2\cos z=e^{i\alpha-\beta}+e^{-i\alpha+\beta}$$

$$2\left(\cosh\frac{z}{n}+\cos\frac{z}{n}\right)=e^{\frac{\alpha+i\beta}{n}}+e^{-\frac{\alpha+i\beta}{n}}+e^{\frac{\beta-i\alpha}{n}}+e^{-\frac{\beta-i\alpha}{n}}$$

all have an infinitely great modulus in general, the exceptional cases are

(1) $\beta=0$, when $\cos z=\pm 1$ by the above choice of $R=k\pi$ or $(k+1)\pi$ and $\cosh\frac{z}{n}+\cos\frac{z}{n}$ is infinite,

(2) $\alpha=0$, when $\cosh z=\pm 1$ and again $\cosh\frac{z}{n}+\cos\frac{z}{n}$ is infinite,

(3) $\alpha=\beta$, when $\cosh\frac{z}{n}+\cos\frac{z}{n}$ is not zero by the above choice of R and $\cosh z$ and $\cos z$ are both infinite.

Thus every element of the integral on the right is zero and therefore $S=\frac{\pi}{8}$.

Question 652.

(S. P. Singaravelu Mudaliar) :—From a point (eccentric angle ϕ) of an ellipse of semi-axes a, b, three normals are drawn to the ellipse ; show that the square on the radius of the circle passing through the feet of the normals is

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

Additional solution by K. B. Madhava.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

Then the equation of the rectangular hyperbola which passes through the feet of the normals from the point (ϕ) is

$$(a^2-b^2)\,xy+b^3\sin\phi\,x-a^3\cos\phi\,y=0\,; \qquad \dots \quad (2)$$

so that eliminating x, we get as the equation giving the ordinates of the four feet of the normals from (ϕ)—one of which of course is ϕ itself,

$$y^4(a^2-b^2)^2+2b^3\sin\phi\,(a^2-b^2)y^3+y^2[(a^4b^2\cos^2\phi+b^6\sin^2\phi$$
$$-b^2(a^2-b^2)^2]-2b^5\sin\phi\,(a^2-b^2)y-b^6\sin^2\phi=0 \qquad \dots \quad (3)$$

Therefore rejecting the root 'ϕ' of this equation, the equation giving the ordinates of the three other points is

$$y^3(a^2-b^2)^2+b\sin\phi\,(a^2-b^2)(a^2+b^2)y^2+b^4(2a^2-b^2)y+$$
$$b^7\sin\phi=0. \qquad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

Now let us suppose that

$$x^2+y^2+2gx+2fy+c=0 \qquad \dots \quad \dots \quad (5)$$

is the circle which passes through these three points. Eliminating x, between (5) and (1), we get as the equation giving the ordinates of these three points and that of a fourth point of intersection

$$y^4(a^2-b^2)^2-4fb^2(a^2-b^2)\,y^3+y^2\,(4f^2b^4+2b^2(c+a^2)(b^2-a^2)+4b^2a^2g^2)$$
$$+4fb^4(c+a^2)\,y+b^4(c+a^2)^2-4g^2a^2b^4=0. \qquad \dots \quad \dots \quad (6)$$

But it is known that this fourth point is diametrically opposite to the foot of the fourth normal which in this case is 'ϕ'.

Multiplying (4) therefore by the factors $(y+b\sin\phi,)$ we get a quartic which can be identified with (6). Making the necessary reductions we have

$$f=-\frac{a^2}{2b}\sin\phi=-\frac{b^2}{2a}\cos\phi\, c=-(a^2+b^2). \tag{7}$$

Hence the equation of the circle (5) is

$$x^2+y^2-\frac{b^2}{a}x\cos\phi-\frac{a^2}{b}y\sin\phi=a^2+b^2.$$

The square on its radius can be put in the form

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

Question 675.

(K. Appukuttan Erady) :—Show that

(1) $\displaystyle\int_0^{\pi}\frac{e^{b\cos\Theta}\cos(b\sin\Theta)}{1-2a\cos\Theta+a^2}\,d\Theta=\frac{\pi e^{ab}}{1-a^2};$

(2) $\displaystyle\int_0^{\pi}\frac{\log(1-2b\cos\Theta+b^2)}{1-2a\cos\Theta+a^2}\,d\Theta=\frac{2\pi}{1-a^2}\log(1-ab).$

Solution by K. B. Madhava.

The question is not rigorously worded; it is obvious that we should have $o<a<1$ in either case, and $o<(b)a<1$ in the second, in addition. There are various ways of establishing these well known results. We will employ the method of contour integration so as to bring out prominently the necessity for the limits of a and b.

Put $z=\cos\Theta$ and corresponding to the interval 0 to 2π for Θ, we shall have as the region of the variable z, a circle (c) of radius (a) about the origin and then the integral

$$(1)=\tfrac{1}{2}\int_0\frac{dz\; e^{bz}}{i\,(1-az)(z-a)}$$

which has a simple pole at $z=a$, if and only if $o<a<1$. (A).

For the residue there

$$\lim_{z\to a}\frac{(z-a)e^{bz}}{i(z-a)(1-az)}=\frac{e^{ab}}{i(1-a^2)}$$

and hence by Cauchy's theorem, the given integral

$$(1)\;=\tfrac{1}{2}\cdot2\pi i\cdot\frac{e^{ab}}{i(1-a^2)}=\frac{\pi e^{ab}}{1-a^2}$$

by equating real parts,

Again, for the second part, the integral

$$=\int_0^{2\pi} \frac{\frac{1}{2}\log(1-2b\cos\theta+b^2.)}{1-2a\cos\theta+a^2.}d\theta$$

which by the same transformation as before

$$=\text{real part of} \int_0 \frac{\log(1-bz).}{i(1-az)(z-a)}dz.$$

which has got as its only pole $z=a$, a simple pole; if (A) is satisfied and in addition $o<ba<1$. (B).

Evaluating the residue as before and applying Cauchy's theorem, the integral in question

$$=\frac{2\pi}{1-a^2}\log(1-ab).$$

We thus see the necessity for conditions (A) & (B).

Question 697.

(S. KRISHNASAWMI AIYANGAR):—The base BC of a triangle is given in magnitude and position. If the intercept on the tangent to the incircle parallel to the base and terminated by the sides is of constant length, find the locus of the vertex.

Solution by Martyn M. Thomas, R. D. Karve and others.

Let k be the constant intercept; then

$$k=r\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$=\frac{r\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=2R\sin A\cdot\tan\frac{B}{2}\tan\frac{C}{2}$$

$$=a.\frac{s-a}{s}.$$

Since k and a are constant, it follows that s is constant, and therefore $b+c$ is constant.

Hence the locus of A is an ellipse with foci at B and C.

Questions 702 and 715.

(MARTYN. M. THOMAS) :—Prove that

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,(\log \sin x) \cos x\,(\log \cos x)\, dx = \frac{12-\pi^2}{48}$;

(ii) $\displaystyle\int_0^{\infty} \frac{\sin x \log (\cos x)}{x} dx = \pi \log \frac{1}{\sqrt{2}}$.

[*N.B.*—Q. 715 is only a repetition of Q. 702 (i).]

Solution by J. M. Bose, M.A., B. Sc., D. Krishnamurti, K. R. Rama Iyer and others.

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \log (\sin x) \cos x \log (\cos x)\, dx.$

The required integral becomes

$$\frac{1}{8}\int_0^1 \log z \log (1-z)\, dz, \text{ on putting } \sin x = z^{\frac{1}{2}}.$$

Now

$$\frac{1}{8}\int_0^1 \log z \log (1-z)\, dz = -\frac{1}{8}\int_0^1 \sum_1^{\infty} \frac{z^n \log z}{n} dz$$

$$= -\frac{1}{8}\left[\sum_1^{\infty} \frac{z^{n+1}}{n(n+1)}\left(\log z - \frac{1}{n+1}\right)\right]_0^1;$$

and it is easy to prove

$$\underset{z=0}{\mathrm{L}}\; z^r \log z = 0.$$

Hence the integral becomes

$$\frac{1}{8}\sum \frac{1}{n(n+1)^2} + \frac{1}{8}\sum_1^{\infty}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)^2}\right)$$

$$= \frac{1}{8}\left(1 - \frac{\pi^2}{6} + 1\right) = \frac{12-\pi^2}{48},$$

Hence $\displaystyle\sum \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \ldots \infty = 1.$

and $\displaystyle\sum \frac{1}{(n+1)^2} + 1 = \frac{1}{1^2} + \frac{1}{2^2} + \ldots \infty = \frac{\pi^2}{6}.$

(ii) $\int_0^\infty \frac{\sin x \log \cos x \, dx}{x}$.

We have

$$\log \cos x = -\log 2 + \sum_1^\infty \frac{(-1)^{n+1}}{n} \cos 2nx.$$

Hence

$$\int_0^\infty \frac{\sin x \log \cos x}{x} dx = -\log 2 \int_0^\infty \frac{\sin x}{x} x$$

$$+ \sum_1^\infty \frac{(-1)^{n+1}}{n} \int_0^\infty \frac{\sin x \cos 2nx}{x} dx$$

$$= -\log 2 \int_0^\infty \frac{\sin x}{x} dx +$$

$$\tfrac{1}{2} \sum^\infty \frac{(-1)^{n+1}}{n} \left[\int_0^\infty \frac{\sin (2n+1)x}{n} dx + \int_0^8 \frac{\sin (1-2n)x}{x} dx \right]$$

$$= -\log 2 \cdot \frac{\pi}{2} + \frac{1}{2} \sum_1^\infty \frac{(-1)^{n+1}}{n} \left[\frac{\pi}{2} + \frac{\pi}{2} \right]$$

$$= -\frac{\pi}{2} \log 2 + \frac{\pi}{2} \sum \frac{(-1)^{n+1}}{n}$$

$$= -\frac{\pi}{2} \log 2 \text{; since} \sum_1^\infty (-1)^{n+1} \cdot \frac{1}{n} = 0.$$

Question 704.

(S. Malhari Rao, B. A.) :—If the sum of a number of three digits and the number formed by reversing the digits be divisible by 37, shew that the sum of all such pairs of number is 480×37.

Solution by K. J. Sanjana and N. S. Rajvade.

If the digits of any number be x, y, z, the sum of this and the number formed by reversing the digits is $101x + 20y + 101z$.

Now $101x + 20y + 101z = 111 ((x+z) - 10 (x+z-2y)$, which will be a multiple of 37 when $x+z=2y$, in which case the sum of this pair of numbers is $222y$,

If $x \neq z$, it is seen that when $x=1$, $z=3$ or 5 or 7 or 9, and y is 2, 3, 4 or 5; when $x=2$, $y=3$, 4 or 5; when $x=3$, $y=4$, 5 or 6, omitting numbers already accounted for. Similarly, when x is 4, 5, 6, 7, y has the values

5 or 6, 6 or 7, 7, 8;

and there are no further numbers. The sum of all the possible values of y being 80, the sum of all the numbers found is $222 \times 80 = 37 \times 480$.

When $x=z$, x, y, z are all equal; in this case we get the numbers (unaltered by reversion of digits) to be 111, 222,......999, whose sum is $3 \times 37 \ (1+2...+9) = 37 \times 135$. The proposer has evidently excluded these from consideration.

Additional solution by K. B. Madhava.

Question 706.

(N. Sankara Aiyar, M.A.) :—If n is any number, show that

$$\frac{1}{(n+1)(n+2)}+\frac{1}{2}\frac{1}{(n+2)(n+3)}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{(n+3)(n+4)}+\dots$$

$$=2^{2n+3}\frac{\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+4)}.$$

Solution by K. Appukuttan Erady, K. B. Madhava and K. J. Sanjana, M.A.

The left hand side is the difference of the two series

$$\frac{1}{n+1}+\frac{1}{2}\frac{1}{n+2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{n+3}+\dots$$

and

$$\frac{1}{n+2}+\frac{1}{2}\frac{1}{n+3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{n+4}+\dots,$$

each of which is absolutely convergent.

Hence the given expression

$$=\int_0^1 \left(x^n+\frac{1}{2}x^{n+1}+\frac{1\cdot 3}{2\cdot 4}x^{n+2}\dots\right)dx$$

$$-\int_0^1 \left(x^{n+1}+\frac{1}{2}x^{n+2}+\frac{1\cdot 3}{2\cdot 4}x^{n+3}\dots\right)dx$$

$$=\int_0^1 x^n(1-x)^{-\frac{1}{2}}dx-\int_0^1 x^{n+1}(1-x)^{-\frac{1}{2}}dx$$

$$=\int_0^1 x^n (1-x)^{\frac{1}{2}} dx = \mathrm{B}\ (n+1,\ \tfrac{3}{2})$$

$$=\frac{\Gamma(n+1)\Gamma(\frac{3}{2})}{\Gamma(n+\frac{5}{2})}. = \frac{1}{2}\frac{\Gamma(n+1\sqrt{\pi}}{\Gamma(n+5/2)}.$$

But

$$\Gamma(n+2)\ \Gamma(n+2+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n+3}}\Gamma(2n+4). \quad \text{(WILLIAMSON, p. 164).}$$

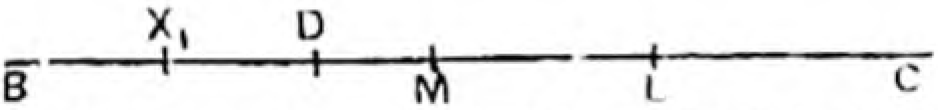
Hence the result.

Question 709.

(R. SRINIVASAN, M. A.) :—The common tangent to the nine points circle of a triangle ABC and the ex-circle opposite to A meets BC in D ; E and F are corresponding points for CA and AB. Show that AD, BE, CF are concurrent.

Solution (1) *by K. J. Sanjana* ; (2) *by K. Appukuttan Erady.*

(1) Let the ex-circle touch BC in X_1 and the nine-points circle in T ; take M the mid-point of BC and L the first of the perpendicular on it from A.



Then $DX_1=DT$ from the ex-circle, and $DT^2=DM.\ DL$ from the nine-points circle. Hence $DX_1^2=DM{\cdot}DL$;

$$\therefore \quad \frac{DX_1}{DM}=\frac{DL}{DX_1}=\frac{X_1L}{X_1M}, \text{ so that } \frac{BD-(s-c)}{\frac{1}{2}a-BD}=\frac{c\cos B-(s-c)}{\frac{1}{2}a-(s-c)}.$$

The right side reduces to $2s/a$, on multiplying by $2a$ above and below ; hence $BD\ (a+2s)=sa+sa-ac=a^2+ab$,

so that
$$BD=\frac{a(a+b)}{2a+b+c}.$$

$$\therefore \quad CD=\frac{a(a+c)}{2a+b+c}, \text{ and } \frac{BD}{CD}=\frac{a+b}{a+c}.$$

Similar results hold for

$$\frac{CE}{AE} \text{ and } \frac{AF}{BF};$$

and the product of the three being unity, AD, BE and CF are concurrent.

If the common tangent of the nine-points circle and the in-circle cuts BC in P, we can show similarly that

$$BP : CP = b-a : c-a.$$

Of course P and the two similar points for CA,, BA, are collinear.

(2) Let the equation to the ex-circle opposite to the angle A be

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+(l\alpha+m\beta+n\gamma)(a\alpha+b\beta+c\gamma)=0,$$

where this meets $\alpha=0$ we must have

$$\frac{\beta}{c(s-b)}=\frac{\gamma}{b(s-c)};$$

Hence

$$\frac{m}{b(s-c)}+\frac{n}{c(s-b)}+\frac{1}{bc}=0 \qquad \ldots \qquad \ldots \quad (1)$$

Again where the circle meets $\beta=0$ we must have

$$\frac{\alpha}{c(s-b)}=\frac{\gamma}{-as};$$

hence

$$\frac{l}{as}-\frac{m}{b(s-c)}+\frac{1}{ab}=0; \qquad \ldots \qquad \ldots \quad (2)$$

and the corresponding equation obtained by finding where the circle cuts $\gamma=0$, *viz.*,

$$\frac{l}{as}-\frac{m}{b(b-c)}+\frac{1}{ab}=0. \qquad \ldots \qquad \ldots \quad (3)$$

Solving these equations we get

$$l=-\frac{s^2}{bc},\quad m=-\frac{(s-c)^2}{ac},\quad n=-\frac{(s-b)^2}{ac}.$$

Now the equation to the nine-points circle can be put in the form

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+(p\alpha+q\beta+r\gamma)(a\alpha+b\beta+c\gamma)=0$$

where $p=\dfrac{a^2-b^2-c^2}{4bc}$, and corresponding values for q and r.

Now the equation to the radical axis of these two circles is

$$(p-l)\alpha+(q-m)\beta+(r-n)\gamma=0;$$

which, when the above values for l,m,n,p,q,r are substituted, reduces to

$$\frac{a\alpha}{b-c}+\frac{b\beta}{a+c}-\frac{c\gamma}{a+b}=0,$$

or in areal co-ordinates

$$\frac{x}{b-c}+\frac{y}{a+c}-\frac{z}{a+b}=0.$$

Hence the common tangent of the nine-point circle and the ex-circle opposite to A divides BC in the ratio of $(a+b):(a+c)$.

Similarly, the other common tangents corresponding to the other ex-circles divide the sides C A, A B in the ratios $b+c:b+a$ and $a+c:a+b$, respectively. The product of three ratios being unity the straight lines joining these points of division to the opposite vertices are concurrent.

QUESTIONS FOR SOLUTION.

ERRATUM.

549. *Corrected* —(K. J. SANJANA, M.A.) :—If

$$S_n^r = 1^r - 2^r + 3^r - \ldots\ldots (-)^{n+1} n^r,$$

prove that

$$S_n^r + \binom{r+1}{1} S_{n-1}^r + \binom{r+1}{2} S_{n-2}^r + \binom{r+1}{3} S_{n-3}^r + \ldots = 0,$$

when r is even and n is any integer equal to or greater than r. [When $n = r-1$, the result is unity.] Shew also how the value of the series may be found when r is odd.

---

757. (S. MALHARI RAO, B. A.) :—Give a solution in positive integers of the equations (1) $x^4 - 6y^2 = 1$, (2) $x^{11} - 17y^2 = x^6$.

758. (S. MALHARI RAO, B. A.) :—If the integers x, y, z represent the sides of a right-angled triangle and x, z are primes greater than 5, shew that y is a multiple of 60 ; and that $x+z=1800$, when $y=1740$.

759. (K. APPUKUTTAN ERADY, M. A.) :—Three circles of radii a, b, c touch one another externally. If the radii of the circles touching them be r, r' $(r<r')$, show that $r, -r'$ are the roots of

$$\rho^2 . \left\{ 2\Sigma\left(\frac{1}{bc}\right) - \Sigma\left(\frac{1}{a^2}\right) \right\} + 2\rho . \Sigma\left(\frac{1}{a}\right) - 1 = 0.$$

760. (K. APPUKUTTAN ERADY, M. A.) :—If

$$f(n) \equiv \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \ldots\ldots$$

show that

$$f(1) + \frac{f(2)}{1!} + \frac{f(3)}{2!} + \ldots\ldots = e.$$

761. (MARTYN, M. THOMAS, M.A.) :—Apply the method of the *Instantaneous Centre* to solve the following :—

At every point of a plane curve a line is drawn making a given angle α with the normal, and let the envelope of these lines be termed the α—evolute. Prove that the β—evolute of the α—evolute of any curve is the α—evolute of the β—evolute of the curve.

[MADRAS, M. A. 1911.]

762. (Professor K. J. SANJANA) :—Two equal uniform rods AB, BC, rigidly jointed at B at an angle 2α, float in a liquid in an unsymmetrical position, with the angle B outside the fluid and A and C immersed : prove that the ratio of the specific gravity of the rods to that of the liquid must lie between

$$\frac{1+\sin^2\alpha}{2} \text{ and } \frac{2\sin^2\alpha}{1+\sin^2\alpha}.$$

If the angle B is immersed and A and C are outside, the limits are

$$\frac{\cos^2\alpha}{1+\sin^2\alpha} \text{ and } \frac{\cos^2\alpha}{2}.$$

763. (S. R. RANGANATHAN, B.A., Hon.) :—A comet is moving along an arc of a parabola, which has the Sun at its focus. Show that the amount of heat it receives from the Sun in any interval is proportional to the angle through which its direction of motion changes during that interval.

764. (S. R. RANGANATHAN, B.A., Hon.) :—Find the form of a solid of revolution, into which a given quantity of matter of uniform density should be put so that its moment of inertia about a diameter of the equatorial plane may be a minimum.

765. (MARTYN M. THOMAS, M.A.) :—Find the simplest Cartesian curve in which, if the abscissae increase by unity, the radii of curvature increase in the ratio of the corresponding slope of the curve.

766. (S. KRISHNASWAMI IYENGAR) :—Find the value of

$$\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots\ldots+\frac{1}{n^n}+\ldots\ldots$$

767. (S. KRISHNASWAMI IYENGAR) :—Show that

$$1-\frac{1}{2^2\cdot5}+\frac{1}{2^4\cdot9}-\frac{1}{2^6\cdot13}\ldots(-)^m\frac{1}{2^{2n}\cdot(4n+1)}\ldots$$
$$=\tfrac{1}{4}\log 5+\tfrac{1}{2}\tan^{-1}2.$$

768. (S. RAMANUJAN) :—If $\psi(x)=\dfrac{x+2}{x^2+x+1}$,

show that

$$\text{(i)}\quad \frac{1}{3}\psi(x^{\frac{1}{3}})+\frac{1}{9}\psi(x^{\frac{1}{9}})+\frac{1}{29}\psi(x^{\frac{1}{27}})+\ldots=\frac{1}{\log x}+\frac{1}{1-x}$$

for all positive values of x; and

$$\text{(ii)}\quad \frac{1}{3}\psi(x^{\frac{1}{3}})+\frac{1}{9}\psi(x^{\frac{1}{9}})+\frac{1}{27}\psi(x^{\frac{1}{27}})+\ldots=-\frac{1}{1-x}$$

for all negative values of x.

769. (S. Ramanujan) :—Show that

$$\log 2\left(\frac{1}{2\log 2\log 4}+\frac{1}{3\log 3\log 6}+\frac{1}{4\log 4\log 8}+\dots\right)$$
$$+\frac{1}{2\log 2}-\frac{1}{3\log 3}-\frac{1}{4\log 4}-\frac{1}{5\log 5}+\dots\dots\dots=\frac{1}{\log 2}.$$

770. (S. Ramanujan) :—If δ_n denote the number of divisors of n (*e.g.*, $\delta_1=1$, $\delta_2=2$, $\delta_3=2$, $\delta_4=3,\dots$), show that

$$\text{(i)}\quad \frac{\delta_1}{1}-\frac{\delta_3}{3}+\frac{\delta_5}{5}-\frac{\delta_7}{7}+\frac{\delta_9}{9}\dots\dots\dots$$

is a convergent series ; and

$$\text{(ii)}\quad \frac{\delta_1}{1}-\frac{\delta_2}{2}+\frac{\delta_3}{3}-\frac{\delta_4}{4}+\frac{\delta_5}{5}\dots\dots\dots$$

is a divergent series in the *strict* sense (*i.e.*, not oscillating).

771. (K. B. Madhava, B. A., Hon.) :—In his solution to Q. 629, Mr. Bhimasena Rao, uses Bromwich P. 447, Ex. 4, (after the method of Dirichlet's integrals), to show that if

$$\psi(t)=\sum_{n=-\infty}^{\infty} e^{-n^2\pi t}$$

we shall have

$$\psi(t)=t^{-\frac{1}{2}}\psi\left(\frac{1}{t}\right).$$

This result is due to Jacobi. See *Ges-Werke.* II, p. 188.

Verify this by integrating $\displaystyle\int \frac{e^{-z^2\pi t}}{e^{2\pi i z}-1}\,dz$ round a rectangle whose vertices are $\pm(\mathrm{R}+\frac{1}{2})\pm i$ and making $\mathrm{R}\to\infty$ in the usual manner.

Show also by the same method, that if $t>0$,

$$\sum_{-\infty}^{\infty} e^{-n^2\pi t-2n\pi at}=t^{-\frac{1}{2}}e^{\pi a^2 t}\left\{1+2\sum_{1}^{\infty} e^{-\frac{n^2\pi}{t}}\cos 2n\pi a\right\},$$

also due to Jacobi.

List of Periodicals Received.

(From 16th March to 15th May 1916.)

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1. Acta Mathematica, Vol. 40, No. 3.
2. American Journal of Mathematics, Vol. 38, No. 1, January 1916.
3. Astrophysical Journal, Vol. 43, No. 1, January 1916.
4. Bulletin of the American Mathematical Society, Vol. 22, No. 5 & 6 February & March 1916.
5. L'Intermediaire des Mathematicians, Vol. 23, Nos. 1 & 2, January & February 1916.
6. Liouville's Journal, Vol. 1, No. 3.
7. Mathematical Gazette, Vol. 8, No. 121, January 1916. (3 Copies).
8. Messenger of Mathematics, Vol.45, Nos 6 & 7, October & November 1915.
9. Monthly Notices of the Royal Astronomical Society, Vol. 6, Nos. 2 & 3, December 1915 and January 1916.
10. Mathematical Questions and Solutions, Vol. 1, Nos. 1, 2, 3 and 4 January, February, March and April 1916. (5 Copies).
11. Philosophical Magazine, Vol. 31, Nos. 183 and 184 [illegible]rch & April 1916.
12. [illegible] .. 24, Nos. 2 and 3, [illegible]bruary and March 1916. (3 Copies)
13. Proceedings of the Royal Society of London, Vol. 92, Nos. 638 and 639, February and March 1916.
14. Quarterly Journal of Mathematics, Vol. 47, No. 1, March 1916.
15. School Science and Mathematics, Vol. 16, Nos. 3 and 4, March and April 1916. (2 Copies).
16. The Tohoku Mathematical Journal, Vol. 9, Nos. 1 & 2, February 1916.
17. American Mathematical Monthly, Vol. 23, Nos. 1 and 2, January and February 1916.

The following books have been received as presents:—Madras University Calendar for 1916, Vols. 1 and 2 and Examination Paper Vol. for 1915.

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Hony. Joint Secretary,

WITH THE CO-OPERATION OF

Prof. R. P. PARANJPYE, M.A., B.Sc. Prof. A. C. L. WILKINSON, M.A., F.R.A.S

and others.

M. A.

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CONTENTS.

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All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. R. KRISHNASWAMI, M.A., 3, Johannes Road, Saidapet, Madras.

Enquiries by intending members and all other communications may be addressed to the Hony. Joint Secretary, D. D. KAPADIA, M.A., B.Sc., 2414, East Street, Poona.

THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. VIII.] AUGUST 1916. [No. 4.

PROGRESS REPORT.

1. The following gentlemen have been elected members at the concessional rate :—

1. *Mr. Sonti Purushotam*, M. A., L. T.—Assistant Professor of Mathematics, Presidency College, Madras, 37, Tholasinga Perumal Street, Triplicane ;

2. *Mr. K. B. Madhava*, B. A. (Honours)—Research Scholars, Madras University, 1/11 Singarachari Street, Triplicane, Madras ;

3. *Mr. C. Krishnamachari*, B. A. (Honours)—Lecturer, Collegiate High School, Mysore.

2. The following books have been received for the Library—

1. *Quartric Surfaces with singular points*—by C. M. Jessop. Camb. University Press, 12s., 1915 ;

2. *Ordinary Differential Equations*—by Dr. J. Morris Page. Macmillon & Co., London, 1897. (Presented by Mr. S. Krishnaswami Aiyangar).

POONA, 31st July 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

Stability and Oscillations of Plane Kites.

By J. M. Bose, M.A., B.Sc.

(*Continued from page* 49.)

6. It is obvious now, that the stability of a plane kite depends entirely on the mode of attachment of the string. If the resultant tension always intersects the axis of symmetry, then some of the equations of the previous articles require slight modification.

Let E′ be the point where the resultant tension intersects the axis of symmetry, H the point of bifurcation, and HP the perpendicular on the y-axis, and let

$$\mathrm{GE}'=h,$$
$$\mathrm{E}'\mathrm{P}=x,$$
$$\mathrm{GP}=b,$$
$$\mathrm{HP}=p.$$

Since $$h+x=b.$$

$$h=b+p\cot\overline{\phi+\chi};$$

so that $$\partial h=-p\operatorname{cosec}^2\overline{\phi_o+\chi}\,\partial\phi=-p'\epsilon^2 \text{ say},$$

where $$p'=p\operatorname{cosec}^2\overline{\phi_o+\chi}=p\sec^2\theta,$$

θ being the angle between the y-axis and the perpendicular to the string and ψ is assumed to be constant for the present.

Thus $$h=h_o-p'\epsilon$$

where $h_o=b-p\tan\theta_o$.

Using this value of h the equations (1), (2), (3), of § 2 remain unaltered, but

$$\mathrm{A}\dot{\theta}_1=-\mathrm{T}(h_o\sin\theta+p'\cos\theta)\epsilon-\eta_o v\mathrm{R}_v-\eta_o w\mathrm{R}_w-\eta_o\theta_1\mathrm{R}_1$$
$$-c\mathrm{R}_o\phi'(\alpha)\left(\frac{w\cos\alpha}{\mathrm{V}}-\frac{v\sin\alpha}{\mathrm{V}}+\theta_1\frac{y\cos\alpha}{\mathrm{V}}\right)\quad\ldots\ (4)'$$

$$\mathrm{B}\theta_2=0\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\ (5)'$$

$$\mathrm{C}\mathring{\theta}_3=-h_o\mathrm{T}(\theta\cos\chi+\chi\sin x)\quad\ldots\quad\ldots\quad\ldots\quad\ldots\ (6)'$$

taking (2), (3), (4)′ and proceeding exactly as before, we get the biquadratic

$$a^4\lambda+b\lambda^3+c\lambda^2+d\lambda+e=0,$$

where

$$a=\mathrm{AM}^2$$

$$b=\mathrm{M}\left[\mathrm{M}\left\{\mathrm{R}_o\left(\frac{c\,\phi'(\alpha)}{\mathrm{V}}y\cos\alpha\right)+\eta_o\mathrm{R}_1\right\}+\mathrm{AR}_w\right]$$

$$c = \mathrm{M}\Big[\mathrm{MT}\,(p'_{,}\cos\theta + h_o\sin\theta) + \frac{c\mathrm{R}_o}{\mathrm{V}}\Big\{\phi'(\alpha)\cos\alpha\,(y\,\mathrm{R}_w - \mathrm{R}_1)\Big\}\Big]$$

$$d = \mathrm{M}\Big[\mathrm{TR}_w(p'\cos\theta + h_o\sin\theta) + a\mathrm{R}_o\,\mathrm{R}_v - \mathrm{R}_o^2\,\frac{\sin\alpha}{\mathrm{V}}\,c\phi'(\alpha)\Big]$$

$$e = -\frac{\mathrm{R}_o^2}{\mathrm{V}}\,c\phi'(\alpha)(\mathrm{R}_w\sin\alpha + \mathrm{R}_v\cos\alpha).$$

These coefficients can be put into different forms by using the different equilibrium conditions; for instance, we have

$$\mathrm{R}_o = \frac{\mathrm{W}h_o}{h_o - \eta_o}\cos\alpha = \mathrm{KSV}^2 f(\alpha)\,;$$

hence multiplying the determinental equation in λ by g^3, we may write e in the form

$$\frac{e}{g^4} = -\frac{h_o}{h_o - \eta_o}\cdot\frac{2\mathrm{W}}{g}\,\mathrm{K}^2\mathrm{S}^2\mathrm{V}^2\,\{f(\alpha)\}^2\,c\phi'(\alpha)\cos\alpha$$

corresponding to Prof. Bryan's (33), *Stability in Aviation*, p. 43.

7. So far we have assumed the length of the string to be infinite. We now proceed to the discussion of the more important case, by taking into account, the variation in the magnitude and direction of the tension, and also assume the length of the string to be finite.

Let H be the point where the string bifurcates into two branches HC, HD; P the foot of the perpendicular from H on the y-axis, and β the angle $\mathrm{H\hat{G}P}$ Also let δ and ϵ be the small increments of χ and ϕ respectively, and $a = \eta_o$

8. *Longitudinal Stability in the case of finite string.*

The roots of the biquadratic given above determine the velocities at the end of any time. We next proceed to show that if we assume the independence of symmetric and asymmetric oscillations, then it is possible to obtain a biquadratic the roots of which determine the co-ordinates of the kite at the end of any time, and hence the nature of the roots of this biquadratic will also determine the conditions for longitudinal *stability of position.*

---

* In Prof. Bryan's memoir α is defined as the inclination of a plane to the x-axis, which is the direction of motion, and θ is the inclination of the direction of motion to a fixed time, namely the horizontal. The symbols therefore correspond respectively to our θ and x.

Let χ and ϕ be the inclination of the string and kite to the vertical, and let r† be the length of the string. The accelerations of the point E′, where the string cuts the axis of symmetry are $(\ddot{r}-r\dot{\chi}^2)$ and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\chi})$, and the accelerations of the centre of gravity relative to this point are $(\ddot{h}-h\dot{\phi}^2)$ and $\frac{1}{h}\frac{d}{dt}(h^2\dot{\phi})$. Hence applying these at the centre of gravity and taking moments about E′, and about the point where the string meets the ground, we get the two equations

$$M\,h\cos\theta\,(\ddot{r}-r\dot{\chi}^2)+M\,h.\,\frac{1}{h}\frac{d}{dt}(h^2\dot{\phi})+M\,k^2\,\ddot{\phi}-M\,h\sin\theta.\,\frac{1}{r}\frac{d}{dt}(r^2\dot{\chi})$$
$$=(h-a)\,R-W\,h\sin\phi \qquad \dots\ (1)$$

$$M\,(h+r\sin\theta)\,\frac{1}{h}\frac{d}{dt}(h^2\dot{\chi})+M\,h\cos\theta\,(\ddot{r}-r\dot{\chi}^2)$$
$$-M\,(r+h\sin\theta)\,\frac{1}{r}\frac{d}{dt}(r^2\dot{\chi})+M\,k^2\ddot{\phi}-M\,r\cos\theta\,(\ddot{h}-h\,\dot{\phi}^2)$$
$$=(h-a+r\sin\theta)\,R-W\,(h\sin\phi+r\sin\chi).\ (2)$$

These are exact equations, but since we are concerned with small motions only, we may reject $\dot{\phi}^2$ and $\dot{\chi}^2$ and regard h, r, $\sin\theta$, $\cos\theta$ on the left hand side of the above equations to be constants.

The variable part of r is $HE'=p\sec\theta$, so that

$$\ddot{r}=\frac{d^2}{dt^2}(p\sec\theta)=p'\sin\theta_0\,\ddot{\theta}=p'\sin\theta_0\,(\ddot{\phi}+\ddot{\chi})$$
$$\dot{r}=p'\sin\theta_0\ \ (\dot{\phi}+\dot{\chi})$$
$$\ddot{h}=-p'(\ddot{\phi}+\ddot{\chi})\ \text{approximately,}$$

where $\qquad p'=p\sec^2\theta_0.$

Introducing these and replacing (2) by the difference of (1) and (2), we have the two equations

$$M(h_0p\ \tan\theta_0+k^2+h_0^2)\ddot{\phi}+M\,h_0(p\tan\theta_0-r_0\sin\theta_0)\ddot{\chi}$$
$$=(h-\eta)R-Wh\sin\phi \quad \dots \qquad \dots\ (1)$$

$$M(h_0\sin\theta_0+p'\cos\theta_0)\ddot{\phi}+M\ddot{\chi}(p'\cos\theta_0-r_0)$$
$$=R\sin\theta-W\sin\chi \qquad \dots \qquad \dots\ (2)$$

---

† The part of the string from the point of bifurcation to the point where it meets the ground is of course assumed to be inextensible and weightless.

We have also for the motion of the centre of gravity

$$M\ddot{\phi}(h+p\tan\theta)+M\ddot{\chi}(p\tan\theta-r\sin\theta)$$
$$=R-W\sin\phi-T\cos\theta. \quad \ldots \qquad \ldots (3)$$

If (3) be multiplied by h and subtracted from (1), we shall of course get (4) of § 2.

If we eliminate $\ddot{\phi}$ and $\ddot{\chi}$ from the above equations, we get

$$\begin{vmatrix} hp\tan\theta_o+k^2+h^2, & (p\tan\theta-r\sin\theta)h, & Wh\sin\phi-R(h-\eta), \\ h+p\tan\theta, & p\tan\theta-r\sin\theta, & W\sin\phi-T\cos\theta-R, \\ h\sin\theta+p'\cos\theta, & p'\cos\theta-r, & W\sin\chi-R\sin\theta, \end{vmatrix}=0.$$

This determinant which vanishes in equilibrium owing to the vanishing of the constituents of the third column, vanishes throughout the small motions which are impressed. To simplify it, we replace the first row, by (first row)$-h$. (second row) and since $p'=p\sec^2\theta$ the factor $(p'\cos\theta-r)$ divides out and the determinant reduces to

$$\begin{vmatrix} k^2 & 0 & \eta R-hT\cos\theta \\ h+p\tan\theta & \sin\theta & W\sin\phi+T\cos\theta-R \\ h\sin\theta+p'\cos\theta & 1 & W\sin\chi-R\sin\theta \end{vmatrix}=0,$$

or

$$T(k^2+h^2\cos^2\theta)=R(k^2+\eta h)\cos\theta-Wk^2(\sin\phi\sec\theta-\tan\theta\sin\chi).$$

This equation gives the tension in terms of the air resistance during motion. It will be noticed that it is (as it should be) independent of the length of the string.‡

To discuss small oscillations we take (1) and (2) and put

$$h=h_o-p'(\epsilon+\delta) \text{ from § 6.}$$
$$\phi=\phi_o+\epsilon$$
$$\eta=a+c\phi'(\alpha)\delta\alpha+\frac{c\phi_1(\alpha)}{V}\theta_1$$
$$\chi=\chi_o+\delta$$
$$R=R_o\delta R=R_o+vR_v+wR_w+\theta_1 R_1$$
$$\theta=\theta_o+\delta\theta=\theta_o+\epsilon+\delta.$$

‡ It was for this relation that I assumed in my previous paper that $T\cos\theta$ or S_x is a function of the same variables as R. Any how this relation shows that the change dT in T owing to a change dR in R is of the first order.

The component velocities of E' are v and $w+h\theta_1$, they are also $\dot{r}$ and $r\dot{\chi}$; hence

$$v=-(r\dot{\chi}\cos\theta+\dot{r}\sin\theta)$$
$$w=r\dot{\chi}\sin\theta-\dot{r}\cos\theta-h_o\dot{\epsilon}.$$

If these values be substituted in (3) of § 2, it will reduce to (1).

Thus we have

$$\delta\alpha=\frac{(w+y\theta_1)\cos\alpha-v\sin\alpha}{V}$$
$$=\frac{(r\sin\overline{\theta+\alpha}-p'\sin\theta\cos\overline{\theta+\alpha})\dot{\delta}+\dot{\epsilon}(y-h_o\cos\alpha-p'\sin\theta\cos\overline{\theta+\alpha})}{V},$$

$$\delta R=r\dot{\chi}\left\{(R_w\sin\theta-R_v\cos\theta)-\frac{p'\sin\theta}{r}(R_w\cos\theta+R_v\sin\theta)\right\}$$
$$+\dot{\epsilon}\left\{(R_1-h\,R_w)-p'\sin\theta(R_v\sin\theta+R_w\cos\theta)\right\}.$$

Hence, substituting, we have the equations of small oscillations

$$\text{(i)}\quad M(h_op\tan\theta_o+k^2+h^2_o)\ddot{\epsilon}+\frac{Mh}{r}(p\tan\theta-r\sin\theta).r\ddot{\delta}$$
$$=r\dot{\delta}\left[(h_o-a)\left\{(R_w\sin\theta-R_v\cos\theta)-\frac{p'\sin\theta}{r}(R_v\sin\theta+R_w\cos\theta)\right\}\right.$$
$$\left.-\frac{cR_o\phi'(\alpha)}{V}.\frac{r\sin\overline{\theta+\alpha}-p'\sin\theta\cos\overline{\theta+\alpha}}{r}\right]$$
$$+\dot{\epsilon}\left[(h_o-a)\left\{(R_1-h_oR_w)-p'\sin\theta(R_v\cos\theta+R_w\sin\theta)\right\}\right.$$
$$\left.-\frac{cR_o\phi'(\alpha)}{V}\left\{(y-h)\cos\alpha-p'\sin\theta\cos\overline{\theta+\alpha}\right\}-\frac{bR_o\phi_1(\alpha)}{V}\right]$$
$$+\epsilon\left[-p'R_o-W(h\cos\phi-p'\sin\phi)\right]+r\delta\left[\frac{p'}{r}(W\sin\phi-R_o)\right];$$

$$\text{(ii)}\quad M(h\sin\theta+p'\cos\theta)\ddot{\epsilon}+M\frac{p'\cos\theta-r}{r}.r\ddot{\delta}$$
$$\dot{\epsilon}\left[\sin\theta\left\{(R_w\sin\theta-R_v\cos\theta)-\frac{p'\sin\theta}{r}(R_w\cos\theta+R_v\sin\theta)\right]\right\}$$
$$+\dot{\epsilon}\left[\sin\theta\left\{(R_1-hR_w)-p'\sin\theta(R_v\sin\theta+R_w\cos\theta)\right\}\right]$$
$$+\epsilon.R_o\cos\theta+r\delta.\frac{R_o\cos\theta-W\cos\chi}{r}$$

To solve this system we assume as usual ϵ, $r\delta$ proportional to $e^{\lambda t}$ and the determinantal equation in λ will be of the type

$$\begin{vmatrix}A\lambda^2+B\lambda+C, & A'\lambda^2+B'\lambda+C' \\ a\lambda^2+b\lambda+c & a'\lambda^2+b'\lambda+c'\end{vmatrix}=0;$$

expanding, we have a biquadratic

$$a_1\lambda^4+b_1\lambda^3+c_1\lambda^2+d_1\lambda+e_1=0.$$

The actual values of the co-efficients can be written down if desired, but the calculation is somewhat laborious; we have however

$$a_1=\frac{M^2}{r}(r-p'\cos\Theta)(k^2+h^2\cos\Theta),$$

$$d_1=(R_v\cos\Theta-R_w\sin\Theta)[W\sin\Theta(p'\cos\alpha-h\sin\alpha)-R_o\cos\Theta(h_o-a)]$$
$$-\frac{cR_o^2}{V}\phi'(\alpha)\cos\Theta\sin(\Theta+\alpha)+\text{terms in }\frac{1}{r},$$

$$e_1=\frac{W\,h\sin\alpha}{r}(W\cos\chi-R_o\cos\Theta)+\frac{W\,p'\cos\Theta}{r}(R_o-W\sin\phi).$$

The roots of this biquadratic give the position of the kite in space at the end of any time, when small variations are made in the value of the co-ordinates. If the conditions of stability given above are satisfied, then the "positional oscillations" will gradually die out. It follows, therefore, that in both cases the fundamental equation in λ is a biquadratic, one root of the former vanishes, when the shift of the centre of pressure is neglected, and one root of the latter vanishes when the string becomes infinitely long.

10. *Interpretation of the Conditions of Stability.* Since a is essentially positive the conditions of stability require that all the other coefficients as well as the discriminant H, where

$$H=b\,c\,d-a\,d^2-e\,b^2$$

should be positive.

If b be equal to zero, *i.e.* if the string be tied to a fixed point on the axis of symmetry which coincides with the centre of gravity, or if the perpendicular from the point of bifurcation to the axis of symmetry passes through the centre of gravity, then since $a=0$ in § 4, all the coefficients except the first two vanish, showing that in this case the stability is dependent on the shift of the centre of pressure.

If b be not zero then we have

$$c=\frac{T\,W^2}{g}.\,b\cos\Theta\,(\tan\Theta+\tan\beta)$$

Now if Θ is positive or numerically less than β, then $\tan\Theta+\tan\beta>0$ since Θ,β are each $<\frac{\pi}{2}$ and $\Theta+\beta<\pi$.

It follows that c will be positive if b be positive, *i.e.*, if the perpendicular from the point of bifurcation intersects the axis of symmetry at a point above the centre of gravity. This condition can be secured by making H C<H D.

We have next

$$d=\frac{W}{g}\left[b\,T\cos^2\theta\,(\tan\theta+\tan\beta)\,R_w+a\,R_oR_v\right].$$

Since $a\,R_o=h_0T\cos\theta=T\cos\theta\,b\,(1-\tan^2\beta\tan\theta)$,

$$d=\frac{W\,T}{g}\,b\cos\theta\left[R_w\tan\overline{\theta+\beta}+R_v\right]\Big(1-\tan\beta\tan\theta\Big);$$

again since $$\beta<\frac{\pi}{2},\ \theta<\frac{\pi}{2}$$

we have $$1\tan\beta\tan\theta>0,\ \text{since}\ \beta+\theta<\frac{\pi}{2}.$$

Hence $d>0$, if $[R_w\tan\overline{\theta+\beta}+R_v]$ has the same sign as b, *i.e.* positive ;

i.e., if $$R_w\left[\tan\overline{\theta+\beta}+\frac{R_v}{R_w}\right]>0.$$

But by § 3, we have

$$\frac{R_v}{R_w}=\frac{2v_0f(\alpha)-f'(\alpha).w_0}{2w_0f(\alpha)+v_0f'(\alpha)}.$$

Now suppose the angle of attack to be so small that the component velocity of air normal to the plane of the kite is negligible; in this case we have

$$\frac{R_v}{R_w}=\frac{2f(\alpha)}{f'(\alpha)},$$

and if we also put $$f(\alpha)=\sin\alpha,$$

then $$\tan\overline{\theta+\beta}+2\tan\alpha>0.$$

The limiting relation at which stability ceases is given by

$$\tan\overline{\theta+\beta}+2\tan\alpha=0.$$

So that in the case of an infinitely long string with forked attachment the critical inclination depends on β.

Proceeding now to the case where the shift of centre of pressure is taken into account, we find that e will be positive if c is positive *i.e.*, if the centre of pressure moves forward as the angle of attack diminishes. This is the case in actual practice.

In the case of a finite string we find the conditions of longitudinal stability far more complicated, and approximations are necessary. If we assume the string to be very long (not necessarily

infinite) so that the terms containing $\frac{1}{r}$ are negligible compared with other terms, we find

$$d_1=(R_w \sin\theta-R_v\cos\theta)[\sin\theta\{Wh_o\cos\phi_o-Wp'\sin\phi_o+p'R_0\}+R_o\cos\theta\,(h-a)]$$

and on using the equilibrium conditions

$$R_o-W\sin\phi_o=T\cos\theta,\ (h_o-a)\,R_o=Wh\sin\phi_o,$$

the above is ultimately reduced to

$$d_1=(R_w\sin\theta-R_v\cos\theta)[pT\tan\theta+Wh\cos\overline{\theta-\alpha}]$$

$$=(R_w\sin\theta-R_v\cos\theta)[Wh_o\cos\overline{\theta-\alpha}+pT\cdot\frac{h-a}{a}\tan\alpha].$$

Since

$$\left.\begin{aligned}T\sin\theta&=W\cos\phi\\ T\cos\theta&=\frac{a}{h_o-a}.W\sin\phi\end{aligned}\right\}\quad\cdots\quad\cdots\quad(1)$$

are conditions of equilibrium; remembering that $\theta=\chi-\alpha$, we have

$$\tan\overline{\psi-\alpha}>\frac{R_v}{R_w};$$

or, for stability, the inclination of the string to the vertical must be greater than $\tan^{-1}\frac{R_v}{R_w}$, or $\tan^{-1}(2\tan\alpha)$ if the angle of attack is small.

But in the case of plane kites with forked attachment as used above, equlibrium with a small angle of attack is not always possible; the least value which the angle of attack can have depends on the length HC.

Now in every position of equilibrium, we have

$$b>h>a$$

and
$$\tan\theta=\frac{h_0-a}{a}\tan\alpha\ \text{ from (1)},$$

so that the condition $\tan\theta-2\tan\alpha>0$ becomes $h_0>3a$.

Hence the smallest value which the angle of attack can have is that in which the centre of pressure just coincides with the foot of the perpendicular from H to the axis of symmetry.

In any case we may conclude that with a long string *positional stability* may exist so long as the inclination of the string to the vertical is greater than $\left(\alpha+\tan^{-1}\frac{R_v}{R_w}\right)$.

For an interesting graphical discussion of this condition we refer the reader to *Stability in Aviation*, pp. 183, 184.

6. *General Expressions for the Resistance Derivatives.* It follows that the nature of the roots of the above biquadratic depends on the value of certain constants called resistance derivatives. The value of the coefficients depends entirely on the particular law of resistance assumed, but a general expression for these can be obtained without any such assumption.

Let $u_o, v_o, w_o, \ldots l_o, m_o, n_o$ be the initial values of the velocity components and the direction of motion of air particles referred to any axes. Let x, y, z be the co-ordinates of an element ds of any surface, and let us assume

$$R_o = f(u_o, v_o, \ldots l_o, m_o, \ldots)$$

and when additional velocity components are impressed we have

$$R_o + \delta R = f(u_o + \delta u, \ldots\ldots l_o + \delta l, m_o + \delta m, \ldots)$$

$$= f(u_o \ldots l_o \ldots) + \frac{\partial f}{\partial u_o}\partial u_o + \frac{\partial f}{\partial v_o}\partial v_o + \frac{\partial f}{\partial w_o}\partial w_o + \ldots$$

Let $V^2 = u^2_o + v_o^2 + w_o^2$

and $\quad U^2 = (Vl_o + u + z\Theta_2 - y\Theta_3)^2 + (Vm_o + v + \ldots)^2 + (Vn_o + w + y\Theta_1 \ldots)^2$

so that $\quad U = V + l_o(u + z\Theta_2 - \ldots) + m_o(v + \ldots) + n_o(w + y\Theta_1 - \ldots)$

approximately

also

$$l = l_o + \delta l = \frac{Vl_o + u + z\Theta_2 - y\Theta_3}{U}$$

$$m = m_o + \delta m = \frac{Vm_o + v + x\Theta_3 - z\Theta_1}{U}$$

$$n = n_o + \delta n = \frac{Vn_o + w + y\Theta_1 - x\Theta_2}{U},$$

so that

$$\delta l = l - l_o = \frac{(m_o^2 + n_o^2)(n - y\Theta_3 + z\Theta_2) - l_o m_o(v + \ldots) - l_o n_o(w + \ldots)}{V}$$

$$\delta m = m - m_o = \frac{(l_o^2 + n_o^2)(v + x\Theta_3 - z\Theta_1) - \ldots\ldots\ldots - m_o n_o(w + \ldots)}{V}$$

$$\delta n = n - n_o = \frac{(l_o^2 + m_o^2)(w + y\Theta_1 - \ldots) - m_o n_o(v + \ldots) - l_o n_o(n + \ldots)}{V}$$

we have also $\delta u = u + z\Theta_2 - y\Theta_3$, $\delta v = v + x\Theta_3 - z\Theta_1$ $\delta w = w + y\Theta_1 - x\Theta_2$.

Substituting these values and arranging, we have

$$\delta R = uR_u + vR_v + wR_w + \Theta_1 R_1 + \Theta_2 R_2 + \Theta_3 R_3$$

$$= u\left[fu_o + fl_o\frac{m_o^2 + n_o^2}{V} - fm_o\frac{l_o m_o}{V} - fn_o\frac{l_o n_o}{V}\right]$$

$$+v\left[fv_o-fl_o\frac{l_om_o}{V}+fm_o\frac{l_o^2+n_o^2}{V}-fn_o\frac{m_on_o}{V}\right]$$

$$+w\left[fw_o+fn_o\frac{l_o^2+m_o^2}{V}-fl_o\frac{l_on_o}{V}-fm_o\frac{m_on_o}{V}\right]$$

$$+\Theta_1\left[yfw_o-zfv_o+fl_o\left(\frac{l_om_o}{V}z-\frac{l_on_o}{V}z\right)\right.$$

$$\left.-fm_o\left(\frac{l_o^2+n_o}{V}z^2+\frac{m_on_o}{V}y\right)+fn_o\left(\frac{l_o^2+m_o^2}{V}y+\frac{m_on_o}{V}z\right)\right]$$

$$+\Theta_2\left[zfu_o-xfw_o+fl_o\left(\frac{m_o^2+n_o^2}{V}z+\frac{l_on_o}{V}x\right)\right.$$

$$\left.+fm_o\left(\frac{m_on_o}{V}-\frac{l_om_o}{V}z\right)-fn_o\left(\frac{l_o^2+m_o^2}{V}x+\frac{l_on_o}{V}z\right)\right]$$

$$+\Theta_3\left[xfv_o-yfu_o-fl_o\left(\frac{m_o^2+n_o^2}{V}y+\frac{l_om_o}{V}x\right)\right.$$

$$\left.+fm_o\left(\frac{l_o^2+n_o^2}{V}x+\frac{l_om_o}{V}y\right)-fn_o\left(\frac{m_on_o}{V}x-\frac{l_on_o}{V}y\right)\right].$$

where $fu_o,\dots fl_o,\dots,$ stand for $\frac{\partial f}{\partial u_o},\dots\frac{\partial f}{\partial l_o},\dots$

The values of R_u, R_v, R_w can now be obtained by multiplying the coefficients of u, v, ... by an element ds of a plane and integrating over the whole plane. If however the breadth of a plane be small compared with the distance from the origin, then the resultant velocity of every element ds may be assumed to be the same, in which case the above expressions will slightly simplify.

For plane kites with axes chosen as above, the terms containing z will vanish; the terms containing x *will also disappear if the y-axis is an axis of symmetry.* Further we have

$$l_o,\ l,\ m,=0,\ m_o=\sin\alpha,\ n_o=\cos\alpha,\ n=1\,;$$

when these values are substituted we find that the coefficients of u, Θ_2, Θ_3 vanish.

In any other case the value of these coefficients can be simplified by a proper choice of axes and the law of resistance. For instance, if we assume that the thrust on an element of a plane is given by

$$dR=KV^2.\sin\alpha.\,ds,$$

we have

$$f(u_o,\ v_o\dots l_o,\ m_o,\dots)=ds\,(u^2_o+v^2_o+w^2_o)(ll_o+mm_o+nn_o)\,;$$

if we also assume the air to blow along the axis of x with a velocity U then we may put after differentiation

$$u_o=U\,;\ v_o,\ w_o,\ m_o,\ n_o=0\,;\ l_o=1\,;$$

we thus find $\quad f u_o = 2\mathrm{KU}l ds, \frac{fl_o}{\mathrm{V}} = \mathrm{KU}l ds \frac{fm_o}{\mathrm{V}} = \mathrm{KU}m ds$, etc.,

and $\mathrm{R}_u = 2\int \mathrm{K}l\mathrm{U}ds,\ \mathrm{R}_v = \int \mathrm{K}m\mathrm{U}ds\ \mathrm{R}_w = \int \mathrm{K}n\mathrm{U}ds$

$$\mathrm{R}_1 = \int \mathrm{KU}(ny - mzds), \text{ etc.}$$

i.e., they reduce to the forms used by Prof. Bryan in *Stability in Aviation*, p. 124.

In the above case we have assumed that the thrust on an element depends on the velocity of that element only, and is independent of the motion of other elements surrounding it. Again a current of air impinging on a rotating lamina cannot be expected to behave in the same way as if the motion of the lamina were of uniform translation. Owing to the difference of velocities between the different parts of the lamina, the total thrust cannot be expected to be the same, and a consideration of these facts led Prof. Byran to assume that the thrust on the lamina is a function, not only of the linear velocities and the directions of motion, but also of the angular velocities. If we introduce θ_1/V, θ_2/V, θ_3/V in the above functional form for R, the effect will be to introduce additional terms $\frac{f_1}{\mathrm{V}}, \frac{f_2}{\mathrm{V}}, \frac{f_3}{\mathrm{V}}$ in the coefficients of $\theta_1, \theta_2, \theta_3$. They are termed "*rotary derivatives*" by Prof. Byran.||

For plane kites let us assume

$$\mathrm{R} = \mathrm{KSV}^2 f(\alpha);$$

we assume that for motion in the $y-z$ plane, $f(\alpha)$ is a function not only of α but also of $\frac{\theta_1}{\mathrm{V}}$.

With this assumption we may immediately deduce the expressions for R_v, R_w, R_1 from the above on putting

$$f(u_o \ldots l_o \ldots n_o) = \mathrm{K}ds(u_o^2 + v_o^2 + w_o^2)\, f\left(\frac{\pi}{2} - \cos^{-1}(ll_o + mm_o + nn_o)\right)$$

where, after differentiation, we may put

$$n_o,\ l_o = 0;\ v_o = \mathrm{V}\sin\alpha,\ w_o = \mathrm{V}\cos\alpha$$

$$m_o = \sin\alpha,\ n_o = \cos\alpha;$$

also $z=0$, and if the y-axis is an axis of symmetry then we may put $x=0$.

[In a recent issue of the *Bulletin* of the Calcutta Mathematical Society (vol. v) there is a note by Mr. B. N. Rau, questioning the

|| See *Stability in Aviation* §§ 24-29.

validity of certain assumptions made in my first paper on this subject For instance one of the equations was

$$M\dot{u} = W\,\theta + S_x$$

where $$S_x = S_x\,(u,\ \theta_2,\ \theta_3).$$

We need not discuss any further why those particular variables, u, θ_2 θ were taken in the x—component of the tension. But Mr. Rau gives (what he considers to be) an example, to prove that if after the elimination of other unknown quantities, the tension is expressed as a function of the velocities, such a function is not necessarily expansible in a series whose terms diminish. His example is as follows:

For small oscillation

$$u = a \sin \omega\, t, \text{ where } a \text{ is small};$$

and if the equation of motion is

$$M\dot{u} = S_x,$$

we get $$S_x = Ma\omega\left(1 - \frac{u^2}{a^2}\right)^{\frac{1}{2}};$$

and since u is of the same order as a the successive terms in the expansion of $a\left(1-\frac{u^2}{a^2}\right)^{\frac{1}{2}}$ are not negligible.

This is of course true. But the fallacy lies in the fact, that the expression $Ma\,\omega\left(1-\frac{u^2}{a^2}\right)^{\frac{1}{2}}$ is deduced from $a \sin \omega t$, which cannot again represent u unless the successive terms in the expansion of the "force function" are negligible. This is a necessary condition for small oscillation. As an example we may take the approximate equation of the pendulum

$$\ddot{\theta} + a^2\theta = 0, \text{ where } a^2 = \frac{g}{l},$$

which leads to

$$\dot{\theta} = \ a\left(1 - \frac{\theta^2}{a^2}\right)^{\frac{1}{2}};$$

and the right hand side cannot be expanded as stated above. But it in easy to show that it is the first term of the expansion of the integral of the exact equation

$$\ddot{\theta} + a^2 \sin \theta = 0.$$

In this paper as well as in my previous papers, I have assumed

$$R = R(v,\ w,\ \theta_1);$$

but Mr. Rau maintains that the "angle of attack" should have been included among the variables on the right hand side. But he has probably overlooked the relation

$$\delta\alpha=\frac{(w+y\theta_1)\cos\alpha-v\sin\alpha}{V}$$

which shows that the angle of attack is not an independent variable and therefore cannot occur explicitly in the function. (See § 8 of this paper ; and §§ 16, 18, 23 and also the last four lines of p. 172 of Prof. Bryan's *Memoir*)].

SHORT NOTES.

The Roots of a Derivative of a Rational Function.

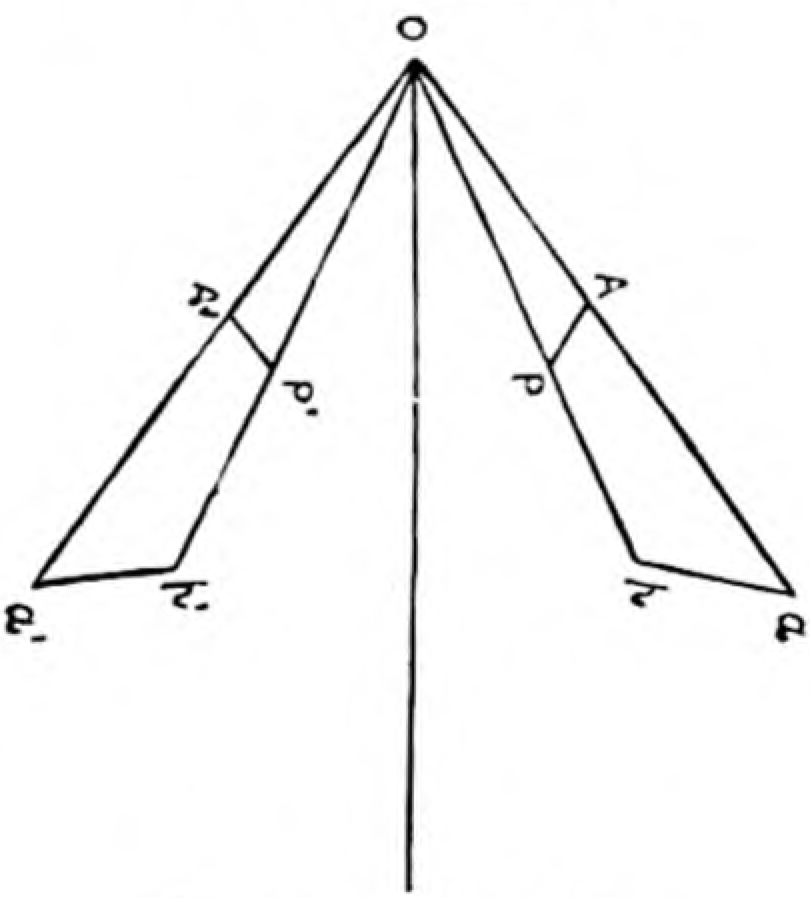
1. MR. L. R. FORD, M.A., contributes a *Note* on the above subject to the Edinburgh Mathematical Society [Vol. XXXIII, 1915, Part 2, p. 103]. The theorem 'that in the complex plane of the variable the smallest convex rectilinear polygon surrounding the roots of a polynomial surrounds also the roots of its first derivative' *, with allied theorems and extensions follows from the properties of the *Harmonic Centre* given by me in this Journal [Vol. IV, 1912, p. 96]. The extension of the above theorem to the more general case of a rational function with a *pole*, discussed by Mr. Ford in the above *Note*, is—

"*If $f(z)$ is a rational function of z whose only pole is at the point a, the smallest circular polygon surrounding the roots of $f(z)$—the sides of the polygon passing through a, and the polygon lying entirely without or entirely within each of its bounding circles—surrounds also the roots of $f'(z)$, with the possible exceptions of two roots at infinity.*"

The method of transformation employed by Mr. Ford virtually amounts to saying that, if $zz'=1=\alpha\alpha'$, then

$$\frac{1}{(z-\alpha)}\equiv\frac{1}{\left(\frac{1}{z'}-\frac{1}{\alpha'}\right)}\equiv\frac{\alpha'z'}{\alpha'-z'}\equiv z'^2\left(\frac{1}{z'}-\frac{1}{z'-\alpha'}\right),$$

which according to *Vector Algebra* signifies that



$$\frac{1}{\overline{AP}}\equiv\overline{Op'}^2\left\{\frac{1}{\overline{Op'}}-\frac{1}{\overline{\overline{a'p'}}}\right\} \quad \ldots \quad \ldots (1)$$

* Osgood: *Lehrbuch der* Funktionen theorie, Vol. I, 1912, p 211; Hayashi, in the *Annals of Mathematics*, Vol. 15, 1914, p 112; Irwin, in the Annals of Mathematics, Vol. 16, 1915, p. 188.

where the points p, a are inverses of P, A with respect to the unit circle and the accented letters denote the reflections of the corresponding unaccented letters in the initial line.

Hence, we deduce the general result

$$\sum\left(\frac{1}{z-\alpha_r}\right)=z'^2\left\{\sum\left(\frac{1}{z'}\right)-\sum\left(\frac{1}{z'-\alpha'_r}\right)\right\} \qquad \ldots (2)$$

for values of r form 1 to n.

Now, let $\phi(z)\equiv f(z)/z^n=(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)/z^n=F(z')$; so that $f(z)=F(z').z^n$. Then since $\phi(z)=F(z')$,

$$\phi'(z)=F'(z').\frac{dz'}{dz}=F'(z')\cdot(-z'^2) \qquad \ldots \qquad \ldots (3)$$

$$\therefore \qquad \frac{\phi'(z)}{\phi(z)}=-z'^2\frac{F'(z')}{F(z')}=\frac{f'(z)}{f(z)}-\frac{n}{z}, \text{ from } (2)$$

$$=\sum\left(\frac{1}{\overline{A_rP}}\right)-\frac{n}{\overline{OP}}. \qquad \ldots \qquad \ldots (4)$$

In other words, the roots of $\phi'(z)$ must satisfy the condition that *the resultant of forces inversely proportional to A_rP is n times the inverse of OP.*

Further, the roots of $\phi'(z)$ correspond to those of $F'(z')$ by (3), and the latter lie within the rectilinear polygon determined by the polynomial $F(z')$. Also, the inverse relation of z and z' shows that the equivalent polygonal boundry for z must be formed by arcs of circles passing through the origin. Hence, the result stated by Mr. Ford.

2. More generally, putting

$$\phi(z)=f(z)/(z-\alpha)^m$$

so that the pole of $\phi(z)$ is α and its roots are the same as those of $f(z)$, we find

$$\frac{\phi'(z)}{\phi(z)}=\frac{f'(z)}{f(z)}-\frac{m}{z-\alpha}$$

$$=\sum\left(\frac{1}{\overline{A_rP}}\right)-\frac{m}{\overline{AP}},$$

where A is the pole.

Hence, in this case, the roots of the derivative $\phi'(z)$ must satisfy the relation

$$\sum\left(\frac{1}{\overline{A_rP}}\right)=\frac{m}{\overline{AP}};$$

that is *the resultant of forces inversely proportional to (A_rP) must be m times the inverse of AP.*

M. T. NARANIENGAR.

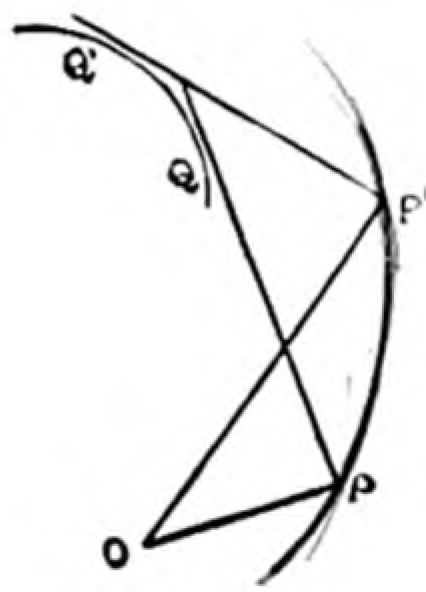
Geometrical representation of a Definite Integral.

A definite integral is usually represented as an *area* in Text-books on the Calculus. It may, however, with equal facility, be expressed as an *arc* as the following method shows:

Consider the integral

$$I=\int_{\alpha}^{\beta} f(\theta)\, d\theta \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

in relation to the curve $r=f(\theta)$, which is the locus of P.



Suppose the *first negative* pedal is the locus of Q. Then approximately

$$r\delta\theta = PQ + QQ' - P'Q' = \delta s' - \delta t', \quad \ldots \quad \ldots \quad (2)$$

denoting PQ by t' and QQ' by $\delta s'$.

Integrating (2), we have

$$\int r d\theta = [s' - t']$$

between proper limits.

In other words, the integral I is represented by an arc of the first negative pedal of $r=f(\theta)$, and its bounding tangents.

Cor. If the curve $r=f(\theta)$ is a closed curve C, the contour integral

$$\int_C r d\theta = C'$$

where C' is the whole arc of the first negative pedal of C.

M. T. Naraniengar.

Note on Question 737.

The importance of the result depends on the extension made which itself presents no difficulty; the definition of the set B_k is in no way dependent on the characters of C, and the quotient $A_k/2k$ can be found for any function that is not negative, for any set of points C, and for any positive value of k. If then $A_k/2k$ tends to a definite limit, this limit may be used to define the linear interval of the function f for the set of points C, and the linear integral of a function g which is sometimes positive and sometimes negative may be defined as $Q-R$, where Q is the linear integral of the function which is equal to g when g is positive and is elsewhere zero, and R is the linear integral of the function which is equal to $-g$ when g is negative and is elsewhere zero. The convergence of $A_k/2k$ in general deserves investigation, and since the method may obviously be extended to the definition of line integrals and of surface integrals in space, a research is suggested which may be difficult. It is to be noticed that the regions and numbers obtained depend only on the form of the set found by completing C (adding to C all of its limiting points which it does not include), and that therefore the definitions may be valuable only in the case of complete sets.

The theorem given affords a simple exercise in the kinematical treatment of differential geometry. As a current point O describes a curve C, the circle with centre O and radius kf describes a band of variable width, and the edges of this band are traced by two points Q, R in which the circle touches the boundaries. In the circumstances described, if k is sufficiently small the region B_k consists of the regions swept by the two lines OQ, OR as O describes the curve, together with sectors corresponding to the end-points if the curve is not closed.

At the current point O, let OT be the tangent, OE the normal making a positive right angle with OT, and κ the curvature; let the current circle meet its envelope in a point P such that the angle EOP is θ; then the co-ordinates of P with respect to OT and OE are $-kf\sin\theta$, $kf\cos\theta$, and therefore the velocity of P with respect to the arc of C has components

$$1-kf'\sin\theta-kf(\kappa+\theta')\cos\theta,\ kf'\cos\theta-kf(\kappa+\theta')\sin\theta.$$

But because the circle touches its envelope, the velocity of P is at right angles to OP, and therefore

$$\sin\theta=kf',$$

and the velocity has components

$$\{\cos\theta-kf(\kappa+\theta')\}\cos\theta,\ \{\cos\theta-kf(\kappa+\theta)\}\sin\theta;$$

thus if α is the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is kf', we may take θ as α for Q and as $\pi-\alpha$ for R.

We note in passing that if kf is numerically greater than unity, the circles have no envelope; indeed, the circle with centre O then contains the neighbouring circles with centres on one side of O and is contained in the neighbouring circles with centres on the other side of O. The case in which kf is numerically equal to unity is interesting as being the case of circles of curvature of a plane curve; as de la Vallée Poussin pointed out although these circles have the curve from which they are derived for a genuine envelope it is entirely false to say that consecutive circles of curvature cut on this envelope. This is a digression, for in the problem with which we are dealing these difficulties are left behind when k has become sufficiently small.

If two points F, G are moving with respect to a parameter t, and if for every value of t their velocities u, v normal to the line through them are positive in the same direction, then if the chord FG is of length l, the rate with respect to t at which this chord sweeps out area is $\frac{1}{2}\, l(u+v)$; this result may be applied in two ways to the present problem. Since Q is moving at right angles to OQ at the rate

$$\cos \alpha - kf\,(\kappa + \alpha'),$$

which is positive if k is sufficiently small (α depends on k, but both α and α' tend to zero with k), and the velocity of O inthe same direction is $\cos \alpha$, the area swept by OQ is

$$\tfrac{1}{2}\int kf\,\{\,2\cos\alpha - kf(\kappa+\alpha')\,\}\,ds\,;$$

similarly the area swept by OR is

$$\tfrac{1}{2}\int kf\,\{\,2\cos\alpha + kf\,(\kappa-\alpha')\,\}\,ds,$$

and if the curve is closed the total area A_k is

$$\int kf\,(2\cos\alpha - kf\alpha')\,ds.$$

If the curve has a beginning and an end, the total area is the sum of this integral and of the area $k^2f^2\,(\frac{1}{2}\pi-\alpha)$ of a sector at the beginning and of the area $k^2f^2\,(\frac{1}{2}\pi+\alpha)$ of a sector at the end—the values of f and α being of course those which correspond to these points. Alternatively with respect to the arc of C the velocities of Q and R at right angles to QR are

$$\{\,\cos\alpha - kf(\kappa+\alpha')\,\}\cos\alpha,\ \ \{\,\cos\alpha + kf\,(\kappa-\alpha')\,\}\cos\alpha,$$

which are positive if k is sufficiently small, and since the length of QR is $2kf\cos\alpha$ the value of A_k is

$$\int 2kf\,(\cos\alpha - kf\alpha')\cos^2\alpha\,ds\,;$$

if C is closed and is the sum of this integral and the areas of segments corresponding to the end-points if C is a curve from one point to a distinct point. The two integrals obtained differ by

$$\int k f (\sin \alpha \sin 2\alpha + k f \alpha' \cos 2\alpha)\, ds,$$

but from the relation of α to f this integral is

$$\tfrac{1}{2}\int \{ d(k^2 f^2 \sin 2\alpha)/ds \}\, ds,$$

which is immediately seen to vanish if C is closed and to represent the difference between the sectors and the segments at the end-points in the more general case. It is interesting to remark that for all sufficiently small values of k the value of A_k is independent of the curvature of the curve the form of the curve affects only the limit below which k must lie for the result to be true.

ERIC H. NEVILLE.

The Face of the Sky for September and October 1916.

The Sun

enters Lebra on September 23 at 3 P. M. and Scorpio on October 23 at 11 P. M..

Phases of the Moon.

	September.			October.		
	D.	H.	M.	D.	H.	M.
First Quarter ...	5	9	57 A. M.	4	4	31 P. M.
Full Moon ...	12	2	1 A. M.	11	12	31 A. M.
Last Quarter ...	19	11	5 A. M.	19	6	39 A. M.
New Moon ...	27	1	4 P. M.	27	2	7 A. M.

The Planets.

Mercury attains its greatest elongation (26° 54′ E) on September 9, is stationary on September 23, is in inferior conjunction on October 5, is stationary on October 14 and attains its greatest elongation (18° 17′ W) on October 21. It is in conjunction with the Moon September 28 and on October 25.

Venus attains its greatest elongation (46° 1′ W) on September 12. It is in conjunction with the Moon on September 23 and on October 23, with Saturn on September 5, with Neptune on September 13 and with ρ Leairs on October 12.

Mars is in conjunction with the Moon on September 2 and September 30 and on October 29.

Jupiter is in opposition to the Sun on October 24. It is in conjunction with the Moon on September 15 and on October 12.

Saturn is in conjunction with the Moon on September 22 at 4-45 A.M. and on October 19.

Uranus is stationary on October 26. It is in conjunction with the Moon on September 9 at 10-30 P. M. and on October 7.

Neptune is in quadrature to the Sun on October 28 and is in conjunction with the Moon on September 22 and on October 20 at 2-16 A.M.

V. RAMESAM.

SOLUTIONS.

Question 495.

(A. C. L. WILKINSON) :—Prove that

$$\begin{vmatrix} 1, & \cos c, & \cos b, & \cos (b-c) \\ \cos c, & 1, & \cos a, & \cos (c-a) \\ \cos b, & \cos a, & 1, & \cos (a-b) \\ \cos (b-c), & \cos (c-a), & \cos (a-b), & 1 \end{vmatrix} = -16\sigma^2$$

where $\sigma = \sin (s-a) \sin (s-b) \sin (s-c)$.

Solution by N. Sankara Aiyar, M.A.

Let l, m, n denote $\cos (s-a)$, $\cos (s-b)$, $\cos (s-c)$ and p, q, r $\sin (s-a)$, $\sin (s-b)$, $\sin (s-c)$, so that $l^2=1-p^2$, $m^2=1-q^2$, $n^2=1-r^2$;

and $\quad \cos a = \cos (s-b+s-c) = mn-qr,$

$\quad \cos (b-c) = \cos (s-c-s+b) = mn+qr.$

The given determinant

$$=1-\Sigma\cos^2 a-\Sigma\cos^2(b-c)+2\Sigma\cos a \cos (a-b) \cos (c-a)$$
$$+\Sigma\cos^2 a\cos^2(b-c)+2 \cos a \cos b \cos c$$
$$-2\Sigma\cos a \cos b \cos(c-a) \cos (b-c)$$
$$=1-\Sigma(lm-pq)^2-\Sigma(lm+pq)^2+2\Sigma(lm-pq)(mn+qr)(ln+pr)$$
$$+\Sigma(l^2m^2-p^2q^2)^2+2\Pi(lm-pq)-2\Sigma(l^2m^2-p^2q^2)(m^2n^2-q^2r^2)$$
$$=1-2\Sigma(l^2m^2+p^2q^2)+\Sigma(l^2m^2-p^2q^2)^2$$
$$-2\Sigma(l^2m^2-p^2q^2)(m^2n^2-q^2r^2)+8(l^2m^2n^2-p^2q^2r^2)$$
$$=1-2\Sigma(1-p^2-q^2+2p^2q^2)+\Sigma(1-p^2-q^2)^2-$$
$$2\Sigma(1-p^2-q^2)(1-q^2-r^2)+8(1-\Sigma p^2+\Sigma p^2q^2-2p^2q^2r^2)$$
$$=1-6+4\Sigma p^2-4\Sigma p^2q^2+3-4\Sigma p^2+\Sigma(p^2+q^2)^2-6$$
$$+8\Sigma p^2-2\Sigma(p^2+q^2)(q^2+r^2)+8-8\Sigma p^2+8\Sigma p^2q^2-16p^2q^2r^2$$
$$=\Sigma(p^2+q^2)^2-2\Sigma(p^2+q^2)(q^2+r^2)+4\Sigma p^2q^2-16p^2q^2r^2$$
$$=2\Sigma p^4+2\Sigma p^2q^2-2\Sigma p^4-6\Sigma p^2q^2+4\Sigma p^2q^2-16p^2q^2r^2$$
$$=-16p^2q^2r^2$$
$$=-16\sigma^2.$$

Question 668.

(J. C. SWAMINARAYAN, M.A.) :—If r and n are integers, prove that the expression

$$1-\frac{2n+1}{1}.{}_rC_1+\frac{(2n+1)(2n+3)}{1.3}{}_rC_2-\frac{(2n+1)\,(2n+3)\,(2n+5)}{1.3.5}.{}_rC_3$$
$$\frac{\ldots\ldots(-1)^r(2n+1)(2n+3)\ldots(2u+2n-1)}{1\cdot3\cdot5\ldots(2r-1)}{}_rC_r$$

is equal to

$$\frac{(-2)^r.{}_nP_r}{1\cdot3\cdot5\ldots(2r-1)}$$

as long as r is not greater than n, but vanishes if $r>n$.

Solution by K. R. Rama Aiyar.

We have

$$(1+x)^{-(n+\frac{1}{2})}=1-\frac{(2n+1)x}{2}+\frac{(2n+1)(2n+3)x^2}{2\cdot4}-\ldots$$

$$+(-)^r\frac{(2n+1)(2n+3\ldots\ldots(2n+2r-1)}{2\cdot4\cdot6\ldots\ldots2r}x^r+\ldots,$$

and

$$(1+x)^{r-\frac{1}{2}}=1+\frac{(2r-1)}{2}x+\frac{(2r-1)(2r-3)x^2}{2\cdot4}+\ldots$$

$$+\frac{(2r-1)(2r-3)\ldots\ldots3\cdot1}{2\cdot4\cdot6\ldots\ldots2r}x^r+\ldots$$

Multiplying the two series we find the coefficient of x^r in the expansion of $(1+x)^{r-n-1}$ is equal to

$$\frac{(2r-1)(2r-3)\ldots3\cdot1}{2\cdot4\cdot6\ldots\ldots2r}-\frac{2n+1}{2}\cdot\frac{(2r-1)(2r-3)\ldots3}{2\cdot4\cdot6\ldots2r-2}+\ldots\ldots$$

$$+\ldots\ldots+(-)^r\frac{(2n+1)(2n+3)\ldots(2n+2r-1)}{2\cdot4\cdot6\ldots\ldots2r}.$$

$$=\frac{(2r-1)(2r-3)\ldots3\cdot1}{2\cdot4\cdot6\ldots2r}\left\{1-\frac{2n+1}{2\cdot1}2r+\frac{(2n+1)(2n+3)}{1\cdot3\cdot2\cdot4}.2r(2r-2)+\ldots\right.$$

$$\left.+\ldots+(-)^r\frac{(2n+1)(2n+3)\ldots\ldots(2n+2r-1)}{1\cdot3\ldots\ldots(2r-1)\,2\cdot4\ldots2r.}\cdot2r(2r-2)\ldots4\cdot2\right\}$$

$$=\frac{(2r-1)(2r-3)\ldots\ldots1}{2^r.r\,!}\left\{1-\frac{2n+1}{1}\cdot{}^rC_1+\frac{(2n+1)(2n+3)}{1-3}{}^rC_2-\ldots+\right.$$

$$\left.+\ldots+(-)^r\frac{(2n+1)(2n+3)\ldots\ldots(2n+2r-1)}{1\cdot3\cdot5\ldots\ldots(2r-1)}{}^rC_r\right\}$$

$$=\frac{1\cdot3\cdot5\ldots\ldots(2r-1)}{2^r.r\,!}.\ \text{S, where S denotes the series given.}$$

But we know that the coefficient of x^r in the expansion of $(1+x)^{r-n-1}$ is

$$(-)^r.\frac{(n-r+1)(n-r+2)\ldots n}{r\,!}$$

if $r\leq n$, and zero if $r>n$.

Hence the result stated.

Question 669.

(J. C. SWAMINARAYAN, M. A.):—Prove that when n is a positive integer,

$$(b^2-a^2)^n+(2n+1)\frac{2n}{2\,!}(b^2-a^2)^{n-1}a^2$$

$$+\frac{(2n+1)(2n+3)(2n)(2n-2)}{4\,!}(b^2-a^2)^{n-2}a^4+\ldots$$

$$=b^{2n}+\frac{(2n)^2}{2\,!}b^{2n-2}a^2+\frac{(2n)^2(2n-2)^2}{4\,!}b^{2n-4}a^4+\ldots$$

Solution by R. D. Karve and K. R. Rama Aiyar.

The coeff. of $b^{2n-2r}a^{2r}$ on the left side

$$=(-1)^r[{}_nC_r-(2n+1)\frac{2n}{2\,!}\cdot{}_{n-1}C_{r-1}$$

$$+(2n+1)(2n+3)\frac{2n(2n-2)}{4\,!}\cdot{}_{n-2}C_{r-2}-\ldots\Big]$$

$$=(-1)^r nC_r[1-(2n+1)\frac{2n}{2\,!}\cdot\frac{n-1\,!}{n\,!}-rC_1$$

$$+(2n+1)(2n+3)\frac{2n(2n-2)}{4\,!}.rC_2\cdot\frac{n-2\,!}{n\,!}-\ldots\Big]$$

$$=(-1)^r{}_nC_r[1-\frac{2n+1}{1}{}_rC_1+\frac{(2n+1)(2n+3)}{1\cdot3}{}_rC_2-\ldots\Big]$$

$$=\frac{2^r\cdot{}_nC_r\cdot{}_nP_r}{1\cdot3\cdot5\ldots(2r-1)}=\frac{2^{2r}.({}_nP_r)^2}{2^r.r\,!\,.\,1.3.5\ldots(2r-1)}$$

$$=\frac{(2n)^2(2n-2)^2\ldots(2n-2r+2)^2}{(2r)\,!}.$$

Hence the result.

Question 670.

(K. J. SANJANA, M. A.):— Prove that

$$\frac{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots}=1+\frac{1}{3\,!}\pi^2\frac{B_1}{2}+\frac{7}{5\,!}\pi^4\frac{B_3}{2^3}+\frac{3_1}{7\,!}\pi^6\frac{B_5}{2^6}+\ldots$$

where B_1, B_3...are the numbers of Bernoulli.

Solution by (1) *K. B. Madhava and* (2) *K. R. Rama Aiyar.*

(1) We have

$$\text{cosec}\ z=\frac{1}{z}+\frac{2(2-1)}{2\,!}B_1z+\frac{2(2^3-1)}{4\,!}B_3z^3+\ldots$$

$$+\frac{2(2^{2n-1}-1)}{(2n)\,!}B_{2n-1}z^{2n-1}+\ldots\text{(Hobson, p. 340)}$$

$$\therefore \int z \operatorname{cosec} z \, dz = z\left\{1+\frac{2(2-1)}{3!}B_1 z^2+\frac{2(2^3-1)}{5!}B_3 z^4+\dots\right.$$
$$\left.+\frac{2(2^{2n-1}-1)}{(2n+1)!}B_{2n-1}z^{2n}+\dots\right\}$$

$$\therefore \int_0^{\frac{1}{2}\pi} z \operatorname{cosec} z \, dz = \frac{\pi}{2}\left\{1+\frac{2(2-1)}{3!}B_1\left(\frac{\pi}{2}\right)^2+\frac{2(2^3-1)}{5!}B_3\left(\frac{\pi}{2}\right)^4\right.$$
$$\left.+\dots+\frac{2(2^{2n-1}-1)}{(2n+1)!}B_{2n-1}\left(\frac{\pi}{2}\right)^{2n}+\dots\right\}$$

$=\frac{\pi}{2}$ times the expression on the right hand side,

and since we know that the denominator on the left hand side is equal to $\frac{\pi}{4}$, the problem reduces to showing

$$\int_0^{\frac{1}{2}\pi} z \operatorname{cosec} z \, dz = 2\sum_0^{\infty}\frac{(-)^n}{(2n+1)^2}.$$

This is a theorem established in Bromwich, p. 289, by applying Borel's rule for uniform summability of non-convergent and asymptotic series.

In fact it is easily with the definition of the term 'uniform summability' that

$$\sin z+\sin 3z+\sin 5z+\dots=\tfrac{1}{2}\operatorname{cosec} z$$

is uniformly summable in an interval $\left(\delta, \frac{\pi}{2}\right)$ where $0<\delta<\frac{\pi}{2}$.

and that $\displaystyle\int_\delta^{\frac{1}{2}\pi} z \operatorname{cosec} z \, dz = 2\int_\delta^{\frac{1}{2}\pi} \mathop{S}_0^{\infty} z \sin(2n+1)z \, dz.$

$$=2 \mathop{S}_0^{\infty}\int_\delta^{\frac{\pi}{2}} z\sin(2n+1)z\,dz.$$

But $\displaystyle\int_\delta^{\frac{1}{2}\pi} z\sin(2n+1)z\,dz=\frac{\delta\cos(2n+1)\delta}{2n+1}-\frac{\sin(2n+1)\delta}{(2n+1)^2}+\frac{(-)^n}{(2n+1)^2}.$

Again $\displaystyle\sum\frac{\cos(2n+1)\delta}{2n+1}=\tfrac{1}{2}\log(\cot\tfrac{1}{2}\delta.)\quad o<\delta<\pi.$

Also since $\displaystyle\left|\frac{\sin(2n+1)\delta}{(2n+1)^2}\right|\leq\frac{1}{(2n+1)^2}$ (a quantity independent of δ)

$$\sum_0^{\infty}\left|\frac{\sin(2n+1)\delta}{(2n+1)^2}\right|\leq\sum_0^{\infty}\frac{1}{(2n+1)^2}\leq\frac{\pi^2}{8}\leq 1{\cdot}23,$$

the series $\sum_{0}^{\infty} \frac{\sin (2n+1)\delta}{(2n+1)^2}$ is uniformly convergent by Weierstrars's M—Test.

Again

$$\lim_{\delta \to 0} \sum_{0}^{\infty} \left[\frac{\delta \cos (2n+1)\delta}{2n+1} - \frac{\sin (2n+1)\delta}{(2n+1)^2} \right] = 0.$$

Hence $\int_0^{\frac{\pi}{2}} z \operatorname{cosec} z dz = 2 \sum_{0}^{\infty} \frac{(-)^n}{(2n+1)^2}$.

Hence the result as stated.

(2) Left-hand member $= \frac{4}{\pi} \int_0^1 \left\{ x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right\} dx$

$$= \frac{4}{\pi} \int_0^1 \frac{\tan^{-1} x}{x} dx = \frac{2}{\pi} \int_0^1 \frac{\sin^{-1} x}{x\sqrt{1-x^2}} dx$$

[J.I.M.S , Vol. VII, p. 140]

$$= \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} z \operatorname{cosec} z \, dz. \text{ [put } z = \sin^{-1} x].$$

The rest as before.

Question 673.

(S. Krishnaswami Aiyengar) :—If the sides of a polygon of n sides subtend equal angles at a point S, then with S as focus two conics can be described, one circumscribing the polygon and the other inscribed in it. Show that the envelope of polars with respect to the inscribed conic of points on the circumconic is a conic having one focus and one directrix coincident with those of the original conics.

Solution by K. R. Rama Aiyar.

We know that a circle can be circumscribed about and another inscribed in a regular polygon of n sides ; these two circles are concentric and the locus of the poles with respect to the circum-circle of tangents to the incircle is another concentic circle. Now reciprocate with respect to a circle with S as centre. Then the polygon reciprocates into another polygon of n sides each of which subtends the same angle at S. Also the circum-circle and the incircle reciprocate into the inscribed conic and the circumconic, both having S for one of their foci and having

a common directrix, since the circles are concentric; and the envelope of polars with respect to the inscribed conic of points on the circum-conic is a conic having S for one of its foci and one directrix the same as that of the two other conics.

Question 678.

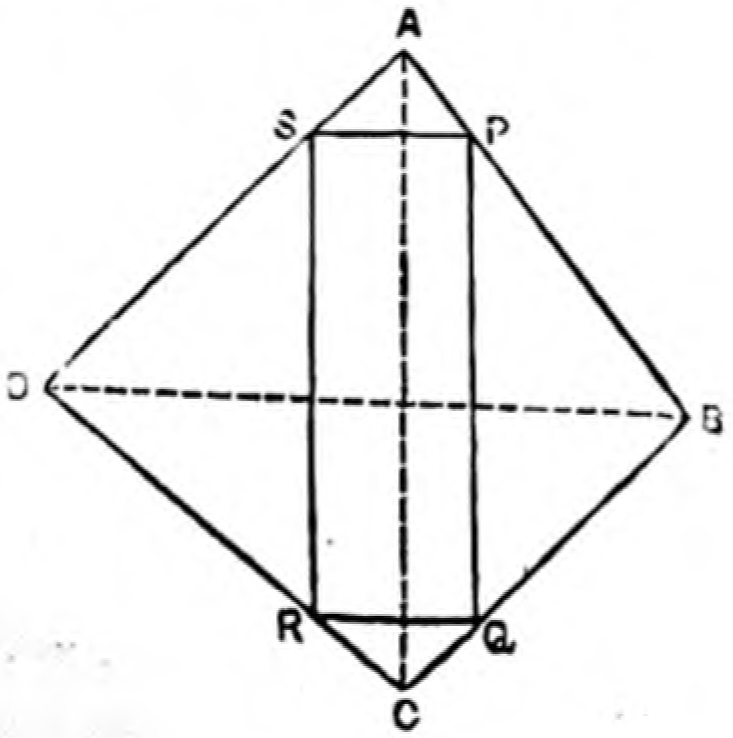
(D. KRISHNA MURTI) :—Show that the centroids of parallelograms inscribed in a skew quadrilateral, so as to have their sides parallel to the diagonals of the quadrilateral, all lie in a straight line which is also the locus of the centres of the conicoids having the sides of the quadrilateral for generators.

Solution (1) *by K. B. Madhava*, (2) *by A. Narasinga Rao.*

(1) By taking the lines joining the mid-points of the sides and of the diagonals as axes, we can obtain the equation to any conicoid as the one passing through the planes ABC and ADC and through ADB and BCD and thus obtain the general equation in the form

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2-\left(\frac{z}{c}-1\right)^2=\lambda\left\{\left(\frac{x}{a}-\frac{y}{b}\right)^2-\left(\frac{z}{c}+1\right)^2\right\}$$

where λ is a variable parameter and by the usual methods available for the purpose the equation to the locus of the centres can be got to be the z-axis. The above form is easily written down by obtaining the equations of the planes from the intercepts they make on the axes.



(2) Let ABCD be the skew quadrilateral and PQRS an inscribed parallelogram.

Consider the centroids of 4 unit masses placed at P, Q, R and S. It is obviously the centroid of PQRS.

The mass at P can be replaced by two masses $\frac{BP}{AB}, \frac{AP}{AB}$. respectively at A and B. Replacing Q, R, S, in the same manner, we have four masses, at A,B, C,D of magnitudes

$$\left(\frac{BP}{AB}+\frac{DS}{AD}\right), \left(\frac{AP}{AB}+\frac{CQ}{BC}\right), \left(\frac{BQ}{BC}+\frac{DC}{CD}\right), \left(\frac{CR}{CD}+\frac{AS}{AD}\right)$$

i.e. $$2.\frac{BP}{AB}, \qquad 2.\frac{AP}{AB}, \qquad 2.\frac{BQ}{BC}, \qquad 2\frac{CR}{CD}$$

i.e. λ, μ, λ, μ, since the sides of the parallelogram are parallel to the diagonals.

Since the masses at A and C are equal as also those at B and D, we may replace the whole system by masses $(2\lambda, 2\mu)$ at the mid-points, O_1, O_2 of AC and BD.

Hence the centroid of parallelogram lies on the line $O_1\, O_2$.

Now let O be the centre of a conicoid passing through the quadrilateral. The tangent planes at A and C meet in BD of which the mid-point is O_2. Therefore OO_2 meets AC.

Similarly OO_1 meets BD. Hence O lies on the line O_1O_2.

N.B.—The condition that the sides of the parallelogram should be parallel to the diagonals is unnecessary.

For, it is readily seen that if a parallelogram is inscribed in a quadrilateral the sides must be parallel to the diagonals, unless the quadrilateral is plane.

Question 688.

(GANPATRAM R. JANI) :—S_r being the sum of the r^{th} powers of the first n natural numbers, prove that

$$S_{r+1}=(r+1)\int^n S_r\, dn+qn.$$

When r is even, $q=0$; when r is odd, q may be found by giving numerical values to n.

Solution (1) *by K. B. Madhava and K. R. Rama Iyer,*

(2) *by R. Srinivasan, M.A. and S. V. Venkatarayasastri, M.A., L.T.*

(1) Denoting by $\phi_n\,(x)$, the Bernoullian polynomial of degree n, *viz*: the coefficient of $\frac{t^n}{n!}$ in the expansion of $t\frac{e^{xt}-1}{e^t-1}$, we easily obtain,

$$\phi_n(x+1)-\phi_n(x)=\text{coefficient of } \frac{t^n}{n!} \text{ in } \frac{t}{e^t-1}\left[e^{(1+x)t}-e^{xt}\right]$$

$$=\text{coefficient in } te^{xt}$$

$$=n\,x^{n-1}.$$

Putting $x=1, 2, \ldots$ in succession

$$1+2^{n-1}+3^{n-1}+\ldots x^{n-1}=\frac{1}{n}\,\phi_n{}^{(x+1)}$$

a well-known result.

Differentiating w. r. t. x,

$$\phi'_n(x)=\text{coefficient of } \frac{t^n}{n!} \text{ in } \frac{t^2 e^{xt}}{e^t-1}$$

$$= \quad ,, \quad \text{ in } t\left[\frac{t(e^{xt}-1)}{e^t-1}+\frac{t}{e^t-1}\right];$$

from which we have

(i) $\quad \phi'_{2m}(x)=2m\phi_{2m-1}(x)$ when $(m>1)$

and (ii) $\phi'_{2m+1}(x)=(2m+1)\{\phi_{2m}(x)+(-)^{m-1}B_m\}$ $(m\geq 1)$.

Thus $\phi'_2(x)=2\phi_1(x)-1$ and all the even $\phi'_{2m}(x)=2m\phi_{(2m-1)}(x)$.

Hence combining the two cases and changing the notation and integrating

$$S_{r+1}=(r+1)\int^n S_r dn+qn.$$

where q is zero when r is even, and when r is odd q is given by (ii).

(2) We have

$$S_r=\frac{n^{r+1}}{r+1}+\tfrac{1}{2}n^r+B_1\frac{r}{2!}n^{r-1}-B_3\frac{r(r-1)(r-2)}{4!}n^{r-3}$$
$$+B_5\frac{r(r-1)(r-2)(r-3)(r-4)}{6!}n^{r-5}-\ldots,$$

the last term containing n or n^2 according as r is even or odd.

[*Vide*: *Higher Algebra* by Hall and Knight, § 406]

$$\therefore\ (r+1)\int^n S_r dn=\frac{n^{r+2}}{r+2}+\tfrac{1}{2}n^{r+1}+B_1\frac{r+1}{2!}n^r-B_3\frac{(r+1)r(r-1)}{4!}n^{r-2}$$
$$+B_5\frac{(r+1)r(r-1)(r-2)(r-3)}{6!}n^{r-4}-\ldots$$

the last term containing n^2 or n^3 according as r is even or odd.

Hence the integral is equal to S_{r+1}, if r is even; and equal to $S_{r+1}\pm$ a term containing n, if n is odd.

Hence the result.

Question 693.

(R. Vythynathaswamy) :—If ABC, A'B'C' be triangles inscribed in the same circle, L_1, L_2, L_3 the latera recta of parabolas having A, B, C for foci and touching the sides of A'B'C' ; L'_1, L_2', L_3' the latera recta of parabolas having A', B', C' for foci and touching the sides of ABC ; show that $L_1.L_2.L_3 = L_1'.L_2'.L_3'$.

Solution by R. Srinivasan, M. A. and K. R. Rama Aiyar.

The pedal line of any point P is the tangent at the vertex of the parabola inscribed in the triangle and having P for its focus. Hence the distance of P from its pedal line is $\frac{1}{4}$ of the latus rectum. If p be this distance and PD the perpendicular to BC, then

$$\frac{p}{PD} = \overline{\text{sine}} \text{ of the angle between the pedal line and BC}$$

$$= \text{sine of angle subtended at the } O^{ce} \text{ by arc AD.}$$

$$\therefore \qquad \frac{p}{PD} \propto \text{chord PA.}$$

$$\therefore \qquad p \propto PA.\ PD.$$

Now PB. PC = PD × (diameter of circle ABC)

$$\therefore \qquad p \propto PA.\ PB.\ PC.$$

$$\therefore \qquad L_1\, L_2\, L_3 = k.\ A'A.\ A'B.\ A'C.\ B'A.\ B'B.\ B.C.\ C'A.\ C'B.\ C'C$$

The symmetry shows that

$$L_1L_2L_3 = L_1'L_2'L_3'.$$

Question 696.

(S. Krishnaswami Aiyangar) :—If λ, μ be the latera recta of the parabola and the rectangular hyperbola of closest contact with a curve at any point, prove that

$$2\lambda\rho = \mu^2$$

Additional solution by K. Appukuttan Erady, M.A.

Since, the parabola and the retangular hyperbola have closest contact with the curve at the point in question, the values of $\rho, \dfrac{d\rho}{ds}$ for these conics must be the same as those for the given curve.

The relation between ρ and $\dfrac{d\rho}{ds}$ for the parabola may be obtained as follows :—

Referred to the axis and tangent at the vertex as axes of co-ordinates the equation to the parabola is $y^2 = \lambda x$.

$$\therefore \qquad y\frac{dy}{dx}=\frac{\lambda}{2}\therefore\frac{dy}{dx}=\frac{\lambda}{2y}$$

$$\therefore \qquad \frac{d^2y}{dx^2}=\frac{-\lambda}{2y^2}\frac{dy}{dx}=\frac{-\lambda}{4y^3}$$

$$\therefore \qquad \rho=\frac{\left(1+\frac{\lambda^2}{4y^2}\right)^{3/2}}{\frac{\lambda^2}{4y^3}}=\frac{(\lambda^2+4y^2)^{3/2}}{2\lambda^2} \qquad \ldots \quad (1)$$

$$\frac{d\rho}{ds}=\frac{1}{2\lambda^2}\times\frac{3}{2}(\lambda^2+4y^2)^{\frac{1}{2}}+8y\,\frac{dy}{ds}$$

$$=\frac{6y}{\lambda^2}(\lambda+4y^2)^{\frac{1}{2}}\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{-\frac{1}{2}}$$

$$=\frac{6y}{\lambda^2}(\lambda^2+4y^2)^{\frac{1}{2}}\left\{1+\frac{4y^2}{\lambda^2}\right\}^{-\frac{1}{2}}$$

$$=\frac{6y}{\lambda}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

From (1) and (2) by eliminating y we get

$$2\lambda^2\rho=\left\{\lambda^2+\frac{\lambda^2}{9}\left(\frac{d\rho}{ds}\right)^2\right\}^{3/2} \qquad \ldots \qquad \ldots \quad (A)$$

The equation to the rectangular hyperbola referred to its as symptotes is $xy=\mu^2/8$, where μ is the latus rectum.

$$\therefore \qquad \frac{dy}{dx}=-\frac{\mu^2}{8x^2},\ \frac{d^2y}{dx^2}=\frac{\mu^2}{4x^3}.$$

$$\therefore \qquad \rho=\frac{\left\{1+\frac{\mu^4}{64x^4}\right\}^{3/2}}{\frac{\mu^2}{4x^3}}=\frac{(\mu^4+64x^4)^{3/2}}{128\mu^2x^3}. \qquad \ldots \quad (\alpha)$$

$$\mu^2\frac{d\rho}{ds}=\left\{-3\frac{(\mu^4+64x^4)^{3/2}}{128x^4}+\frac{3}{2}\frac{(\mu^4+64x^4)^{\frac{1}{2}}}{128x^3}\times 64x4x^3\right\}\frac{dx}{ds}$$

$$i.e., \qquad \frac{\mu^2 d\rho}{ds}=3\left\{(\mu^4+64x^4)^{\frac{1}{2}}-\frac{(\mu^4+64x^4)^{3/2}}{128x^4}\right\}\left\{1+\frac{\mu^4}{64x^4}\right\}^{-\frac{1}{2}}$$

$$=3\times 8x^2\left\{1-\frac{\mu^4+64x^4}{128x^4}\right\}=3.\frac{64x^4-\mu^4}{16x^2}.$$

$$\therefore \qquad \frac{16}{3}\mu^2\frac{d\rho}{ds}=64x^2-\frac{\mu^4}{x2};\ \text{and from } (\alpha),\ (128\,\mu^2\rho)^{2/3}=64x^2+\frac{\mu^4}{x^2}.$$

$$\therefore \qquad (128\mu^2\rho)^{\frac{4}{3}}-\frac{256}{9}\left(\frac{d\rho}{ds}\right)^2=256\mu^4 \quad \ldots \qquad \ldots \qquad \ldots \quad (B)$$

Eliminating $\frac{d\rho}{ds}$ from (A) & (B), we have

$$\underline{2\lambda\rho=\mu^2.}$$

Question 700.

(S. RAMANUJAN) :—Sum the series

$$(a+b+1)\left(\frac{a}{b}\right)^2+(a+b+3)\left\{\frac{a(a+1)}{b(b+1)}\right\}^2$$
$$+(a+b+5)\left\{\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\right\}^2+\dots \text{ to } n \text{ terms.}$$

Solution by (1) *K. R. Rama Aiyar*, (2) *by K. Appukuttan Erady.*

(1) Euler's Identity gives

$$(1-a_1)+a_1(1-a_2)+a_1a_2(1-a_3)+\dots$$
$$+a_1a_2\dots a_n(1-a_{n+1})=1-a_1a_2\dots a_{n+1}.$$

In this put

$$a_1=a^2,\ a_2=\left(\frac{a+1}{b}\right)^2,\ a_3=\left(\frac{a+2}{b+1}\right)^2,\quad a_{n+1}=\left\{\frac{a+n}{b+n-1}\right\}^2.$$

Then

$$1-a^2+a^2\left\{1-\left(\frac{a+1}{b}\right)^2\right\}+\frac{a^2}{b^2}(a+1)^2\left\{1-\left(\frac{a+2}{b+1}\right)^2\right\}$$
$$+\dots+\left\{\frac{a(a+1)(a+2)\dots(a+n-2)}{b(b+1)(b+2)\dots(b+n-2)}\right\}^2(a+n-1)^2\left\{1-\left(\frac{a+n}{b+n-1}\right)^2\right\}$$
$$=1-\frac{a^2}{b^2}\left(\frac{a+1}{b+2}\right)^2\left(\frac{a+1}{b+2}\right)^2\dots\dots\left(\frac{(a+n-1)}{b+n-1}\right)^2(a+n)^2.$$

i.e., $$(1-a^2)+\frac{a^2}{b^2}(a+b+1)(b-a-1)+\left\{\frac{a(a+1)}{b(b+1)}\right\}^2(a+b+3)(b-a-1)$$
$$+\dots+\left\{\frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}\right\}^2(a+b+2n-1)(b-a-1)$$
$$=1-\left\{\frac{a}{b}\cdot\frac{a+1}{b+1}\cdot\frac{a+2}{b+2}\dots\dots\frac{a+n-1}{b+n-1}\cdot(a+n)\right\}^2.$$

Hence the given series S is equal to

$$\frac{1}{b-a-1}\left\{a^2-\left[\frac{a(a+1)\dots\dots(a+n-1)(a+n)}{b(b+1)\dots\dots(b+n-1)}\right]^2\right\}.$$

(2) Let $u_r=(a+b+2r-1)\left\{\dfrac{a(a+1)\dots(a+r-1)}{b(b+1)\dots(b+r+1)}\right\}^2.$

Take the auxiliary series whose $v_r=\left\{\dfrac{a(a+1)\dots(a+r-1)}{b(b+1)\dots(b+r-2)}\right\}^2.$

Then $v_r-v_{r+1}=\left\{\dfrac{a(a+1)\dots(a+r-1)}{b(b+1)\dots(b+r-1)}\right\}^2\left\{(b+r-1)^2-(a+r)^2\right\}$

$$=(b-a-1)u_r.$$

$\therefore\qquad v_1-v_{n+1}=(b-a-1)\mathrm{S}.$

Hence $$\mathrm{S}=\frac{1}{b-a-}\left\{a^2-\left(\frac{a(a+1)\dots(a+n)}{b(b+1)\dots(b+n-)}\right)^2\right\}.$$

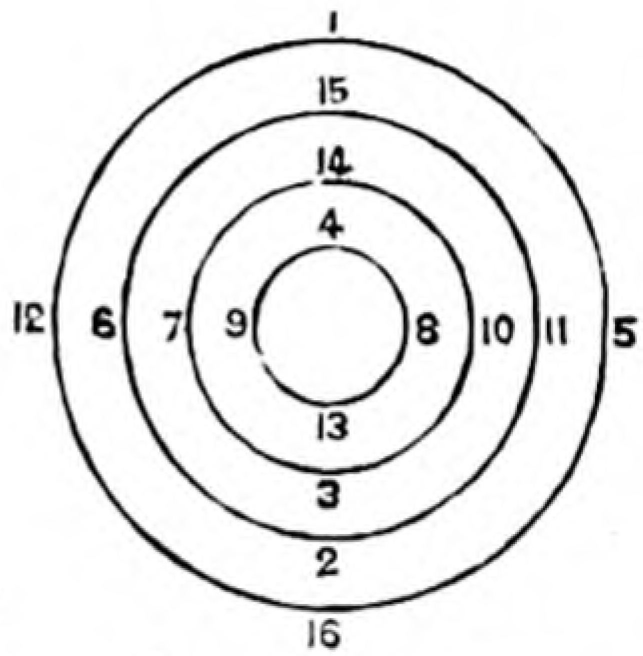
Question 705.

(S. Malhari Rao, B.A.) :—The circumferences of four concentric circles are cut by two diameters. Arrange the numbers 1, 2, 3,...16 at the points of intersection in such a way that the sum of the numbers on each radius and on each circumference may be the same, and also the sum of every pair of diametrically opposite numbers may be the same.

Solution by R. D. Karve and others.

The usual magic square

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16



suggests the accompanying arrangement. Of course the arrangement may be varied.

Question 712.

(J. C. Swaminarayan, M.A.) :—If $a > b$ and

$$f(a, b) = \int_0^\pi \log(a + b\cos\theta)\, d\theta,$$

prove that

$$f(a, b) = \frac{1}{2} f\left(a^2 - \frac{b^2}{2}, \frac{b^2}{2}\right) = \pi \log\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right).$$

Solution by (1) D. Krishnamurti, L. N. Subramanyam and K. B. Madhava, (2) by R. Srinivasan, M.A. and (3) by G. A. Kamekar.

(1) Now $f(a, b) = \int_0^\pi \log(a + b\cos\theta)\, d\theta$

$$= \int_0^\pi \log(a - b\cos\theta)\, d\theta \text{ (changing } \theta \text{ to } \pi - \theta).$$

$$\therefore \qquad 2f(a\,b)=\int_0^\pi \log(a^2-b^2\cos^2\theta)\,d\theta$$

$$=\int_0^\pi \log\left(a^2-\frac{b^2}{2}-\frac{b^2}{2}\cos 2\theta\right)d\theta,$$

changing 2θ to $\pi-\theta$, we get

$$=\int_0^\pi \log\left(a^2-\frac{b^2}{2}+\frac{b^2}{2}\cos\theta\right)d\theta$$

$$=f\left(a^2-\frac{b^2}{2},\ \frac{b^2}{2}\right).$$

Again $f(a,b)=\pi\log a+\int_0^\pi \log\left(1+\frac{b}{a}\cos\theta\right)d\theta.$

$$=\pi\log a\cos^2\frac{1}{2}\alpha+\int_0^\pi \log(1+te^{i\theta})(1+te^{-i\theta})\,d\theta$$

putting $\qquad \frac{b}{a}=\sin\alpha=\frac{2t}{1+t^2}$, where $t=\tan\frac{1}{2}\alpha$

$$=\pi\log a\cos^2\tfrac{1}{2}\alpha+\int_0^\pi\left(te^{i\theta}-\frac{t^2}{2}e^{2i\theta}+-\dots\right)$$

$$\times\left(te^{-i\theta}-\frac{t^2}{2}e^{-2i\theta}+\dots\right)d\theta$$

$$=\pi\log a\cos^2\tfrac{1}{2}\alpha+\int_0^\pi 2\left(t\cos\theta-\frac{t^2}{2}\cos 2\theta+-\dots\right)d\theta$$

$$=\pi\log a\cos^2\tfrac{1}{2}\alpha+2\Big[-t\sin\theta+\frac{t^2}{2^2}\sin 2\theta^2+\dots\dots\Big]_0^\pi$$

$$=\pi\log\frac{a}{2}\cdot\left(1+\frac{\sqrt{a^2-b^2}}{a}\right)$$

$$=\pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$$

$$\therefore\ f(a,b)=\tfrac{1}{2}f\left(a^2-\frac{b^2}{2},\frac{b^2}{2}\right)=\pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$$

(2) $\int_0^\pi \log(1+n\cos\theta)\,d\theta=\pi\log\frac{1+\sqrt{1-n^2}}{2}$

(page 177, Roberts' *Integ. Cal.*).

In this, putting $\frac{b}{a}$ for n, we get the required result.

Also

$$f(a, b) = \pi \log \frac{a+\sqrt{a^2-b^2}}{2}.$$

$$\therefore f\left(a^2-\frac{b^2}{2}, \frac{b^2}{2}\right) = \pi \log \left\{ \frac{a^2-\frac{b^2}{2}+\sqrt{a^4-a^2b^2}}{2} \right\}$$

$$= \pi \log \frac{2a^2-b^2+2a\sqrt{a^2-b^2}}{4}$$

$$= \pi \log \left(\frac{a+\sqrt{a^2-b^2}}{2}\right)^2$$

$$= 2f(a, b).$$

(3) Let $I = f(a,b) = \int_0^\pi \log (a+b\cos\theta)d\theta,\ a>b$;

$$J = f\left(a^2-\frac{b^2}{2}, \frac{b^2}{2}\right) = \int_0^\pi \log\left(a^2-\frac{b^2}{2}+\frac{b^2}{2}\cos\theta\right)d\theta.$$

Differentiating I and J with respect to a, we get

$$\frac{\partial I}{\partial a} = \int_0^\pi \frac{1}{a+b\cos\theta}d\theta = \left[\frac{2}{\sqrt{a^2-b^2}}\tan^{1}\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right]_0^\pi$$

$$\frac{\partial J}{\partial a} = \left[\frac{4}{\sqrt{a^2-b^2}}\tan^{1}\sqrt{\frac{a^2-b^2}{a^2}}\tan\theta/2\right]_0^\pi$$

$$\therefore \qquad \frac{\partial I}{\partial a} = \frac{\pi}{\sqrt{a^2-b^2}};\ \text{and}\ \frac{\partial J}{\partial a} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$\therefore$ $I = \pi(\log a+\sqrt{a^2-b^2})+c$; and $J = 2\pi\log(a+\sqrt{a^2-b^2}+c')$.

Now both I and J vanish if $a=1$ and $b=0$.

$$\therefore \qquad c = -\pi\log 2 \text{ and } c' = -2\pi\log 2.$$

Hence

$$I = \pi\log\frac{a+\sqrt{a^2-b^2}}{2}, \text{ and } J = 2\pi\log\frac{a+\sqrt{a^2-b^2}}{2}$$

$$\therefore \qquad I = \tfrac{1}{2}J = \pi\log\frac{a+\sqrt{a^2-b^2}}{2}.$$

Question 716.

(S. Krishnaswami Aiyangar, B.A.) :—Show how to find the sum of

$$1+\frac{a}{1!}+\frac{2^r a^2}{2!}+\frac{3^r a^3}{3!}+\ldots\frac{p^r a^p}{p!}+\ldots$$

Remarks by K. B. Madhava, R. Srinivasan, M.A., and S. R. Ranganathan.

It is quite easy to see that the sum of this is

$$1+\left(a\frac{d}{da}\right)^r e^a.$$

The proposer is referred to p. 184 of the Journal for October 1913 for more general results of this kind.

Question 717.

(C. KRISHNAMACHARY) :—Show that:

$$\phi(x+h)-\phi(x+3h)+\phi(x+5h)-\ldots$$
$$=\phi(x-h)-\phi(x-3h)+\phi(x-5h)-\ldots$$

Solution (1) *by K. B. Madhava and D. G. Dandekar*, **B.Sc**; (2) *by R. Srinivasan, M.A.*

(1) Adopting the method of Edwards' *Diffl. Calc.* § 553, we take the result $\sin\theta-\sin 3\theta+\sin 5\theta\ldots=0$

i.e. $$e^{i\theta}-e^{3i\theta}+e^{5i\theta}-\ldots\equiv e^{-i\theta}-e^{-3i\theta}+e^{-5i\theta}-\ldots$$

and write for $e^{i\theta}$, the operator $e^{h\frac{d}{dx}}=\mathrm{E}h$, when we get

$$\mathrm{E}^h-\mathrm{E}^{3h}+\mathrm{E}^{5h}-\ldots\equiv\mathrm{E}^{-h}-\mathrm{E}^{-3h}+\mathrm{E}^{=5h}-\ldots$$

applying this to operate upon $\phi(x)$, we at once have the desired result.

(2) We know that $\dfrac{x}{1+x^2}=\dfrac{x^{-1}}{1+x^{-2}}$.

In this put $x=e^{h\frac{d}{dx}}=\mathrm{E}^h$ and let both sides operate on $\phi(x)$, then

$$(1+\mathrm{E}^{2h})^{-1}\phi(x)=\mathrm{E}^{-h}(1+\mathrm{E}^{-2h})\mathrm{E}^{-1}\phi(x)$$

i.e. $$\phi(x+h)-\phi(x+3h)+\ldots=\phi(x-h)-\phi(x-3h)+\ldots$$

Remarks by K. B. Madhava.

Expanding in powers of h, by Taylor's theorem we see that the even powers of h agree and the question amounts to showing that the coefficients of the odd powers of h, *viz* :

$$1^{2s+1}-3^{2s+1}+5^{2s-1}-\ldots\text{vanish.}$$

From the fact that the integral $\int_0^\infty e^{-t}u_p(xt)\,dt$ converges uniformly if $\cos\theta\leq 1-a$ (in the notation of Bromwich, §§ 109, 110), we infer that the results for $\displaystyle\mathop{\mathrm{S}}_1^\infty \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta$ can be differentiated any number of times, whence we obtain the results :

$$\mathop{\mathrm{S}}_1^\infty n^{2s+1}\sin n\theta=0,\text{ if }\theta\text{ is not an even multiple of }\pi.$$

$$\mathop{\mathrm{S}}_1^\infty (-)^{n-1}\, n^{2s} \sin n\theta = (-)^s \frac{d^{2s}}{d\theta^{2s}}\left(\tfrac{1}{2}\tan\frac{\theta}{2}\right), \text{ if } \theta \text{ is an odd } \pi\,;$$

from which we have

$$1^{2s+1} - 3^{2s+1} + 5^{2s+1} - \ldots = 0.$$

and

$$1^{2s} - 3^{2s} + 5^{2s} - \ldots = \tfrac{1}{2}(-)^s\, E_s;$$

where E is the Euler number.

Question 728.

(K. Appukuttan Erady, M. A.) :—If $u \equiv (a\ b\ c\ f\ g\ h)(x\ y\ z)^2$, show that $\iiint u^n\, dx\, dy\, dz$ taken throughout the space bounded by $u=1$, is $\dfrac{4\pi}{2n+3}\Delta^{-\frac{1}{2}}$, where Δ is the discriminant of u.

Solution by Martyn M. Thomas, K. V. A. Sastri and C. Bhaskaraiya.

Since $u=1$ represents a central conicoid, transforming the equation to the standard form $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 1$, by turning the co-ordinate axes so as to coincide with the axes of the conicoide, we find $\lambda_1\lambda_2\lambda_3 = \Delta$.

Now consider a concentric, similar ellipsoidal shell

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = h$$

over the surface of which u has the constant value h.

Extended over the *volume* of $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = h$, included in the first octant,

$$\iiint dx\, dy\, dz = \frac{\left(\frac{h}{\lambda_1}\right)^{\frac{1}{2}}\cdot\left(\frac{h}{\lambda_2}\right)^{\frac{1}{2}}\left(\frac{h}{\lambda_3}\right)^{\frac{1}{2}}}{2\times2\times2}\,\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^3}{\Gamma\left(1+1\frac{1}{2}\right)} = \frac{h^{\frac{3}{2}}}{6\Delta^{\frac{1}{2}}}.$$

The differential of this gives the value of $\iiint dx\, dy\, dz$ extended all over the *surface* of the shell, *viz.* $\dfrac{\pi}{4\Delta^{\frac{1}{2}}}\, h^{\frac{1}{2}}\, dh.$

$$\therefore\ \iiint u^n\, dx\, dy\, dz = \frac{\pi}{4\Delta^{\frac{1}{2}}}\int h^n.\, h^{\frac{1}{2}}\, dh,$$ since $u=h$ all over the surface of the shell.

Hence the value of $\iiint u^n\, dx\, dy\, dz$ throughout the solid

$$= 8\,\frac{\pi}{4\Delta^{\frac{1}{2}}}\int_0^1 h^{n+\frac{1}{2}}\, dh$$

$$= \frac{4\pi}{\Delta^{\frac{1}{2}}}\cdot\frac{1}{2n+3}.$$

QUESTIONS FOR SOLUTION.

772. (T. Rajarama Rao, B.A., B.L.) :—To construct a triangle having given the sum of the sides, the vertical angle, and the area of the rectangle formed by the two segments into which the base is divided by the external bisector of the vertical angle.

Wanted a *purely geometrical* solution.

773. (K. Appukuttan Erady, M.A.):—The conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ circumscribes ABC, the triangle of reference. If ρ_1, ρ_2, ρ_3 be the radii of curvature of the conic at A, B, C respectively, show that

$$a^3\rho_1 : b^3\rho_2 : c^3\rho_3 :: (m^2+n^2-2mn\cos A)^{\frac{3}{2}} : (l^2+n^2-2ln\cos B)^{\frac{3}{2}}$$
$$: (l^2+m^2-lm\cos C)^{\frac{3}{2}}$$

774. (N. P. Pandya) :—Find five sets of two numbers each such that their sum is a perfect square, and the sum of their cubes is a perfect fifth power.

775. (K. Srinivasan) :—Prove geometrically

$$1+\mathrm{dn}\,2z=\frac{2\mathrm{dn}^2 z}{1-k^4\,\mathrm{sn}^4 z}.$$

776. (K. Srinivasan) :—Expand in a Fourier Series $\mathrm{cn}^7 z$.

777. (S. Krishnaswami Iyengar) :—Shew that the envelope of the axes of the parabolas having double contact with the ellipse $x^2/a^2+y^2/b^2=1$ at the ends of a focal chord is

$$4a^2(x^2+y^2)^2+27b^4e^6x^2-48ab^2e^2x^2y=0.$$

778. (S. Malhari Rao, B. A.) :—Solve in positive integers $x^2-y^2=4z^2$, $z^2-w^2=15w$, $x+y+z+w=192$.

779. (S. Malhari Rao, B. A.) :—ABC and ADC are two rational triangles. If their areas are rational but unequal, and if BC = CD = 13 inches, and AC = 37 inches, shew that ABCD is a cyclic quadrilateral and that the angle BAD is bisected by AC.

780. (Martyn M. Thomas) :—Solve the difference equation

$$(1+x)u_{x+2}-2e^{2x+1}.x.u_{x+1}+e^{4x}(x-1)u_x=e^{1+x+x^2}$$

781. (Selected) :—Two vectors OP, OQ of the curve

$$r=2a\cos^3\left(\frac{\pi}{4}+\frac{\theta}{3}\right)$$

are drawn equally inclined to the linitial line. Prove that, if s be the length of the arc intercepted, the area included betweeen the curve and the radii is

$$\frac{5as}{8}-\frac{9a^2}{16}\sin\left(\frac{2s}{3a}\right).$$

782. (Selected) :—If S_r is the sum of the squares of the reciprocals of the first r odd numbers, prove that

$$\frac{S_1}{3^2}+\frac{S_2}{5^2}+\frac{S_3}{7^2}+\ldots=\frac{\pi^4}{384}.$$

783. (S. Ramanujan) :—If $x=y^n-y^{n-1}$,

and
$$J_n=\int_0^1 \frac{\log y}{x}dx,$$

show that

(i)
$$J_0=\frac{\pi^2}{6};\ J_{\frac{1}{2}}=\frac{\pi^2}{10};\ J_1=\frac{\pi^2}{12}\ J_2=\frac{\pi^2}{15}.$$

(ii)
$$J_n+J_{\frac{1}{n}}=\frac{\pi^2}{6}.$$

784. (S. Ramanujan) :—If $F(x)$ denotes the fractional part of x (*e.g.* $F(\pi)=.14159\ldots$) and N is a positive integer, show that

(i) $\lim\limits_{N\to\infty} NF(N\sqrt{2})=\frac{1}{2\sqrt{2}};\ \lim\limits_{N\to\infty} NF(N\sqrt{3})=\frac{1}{\sqrt{3}};$

$\lim\limits_{N\to\infty} NF(N\sqrt{5})=\frac{1}{2\sqrt{5}};$

$\lim\limits_{N\to\infty} NF(N\sqrt{6})=\frac{1}{\sqrt{6}};\ \lim\limits_{N\to\infty} NF(N\sqrt{7})=\frac{3}{2\sqrt{7}}.$

(ii) $\lim\limits_{N\to\infty} N(\log N)^{1-p}F\left(Ne^{\frac{2}{n}}\right)=0,$

where n is any integer and p is any positive number.

(iii) In (ii) show that p cannot be zero.

785. (S. Ramanujan) :—Show that

$$\sqrt[3]{3(\sqrt[3]{a^3+b^3}-a)\ (\sqrt[3]{a^3+b^3}-b)}=\sqrt[3]{(a+b)^2-\sqrt[3]{a^3-ab+b^3}}.$$

This is analogous to

$$\sqrt{2(\sqrt{a^2+b^2}-a)(\sqrt{a^2+b^2}-b)}=a+b-\sqrt{a^2+b^2}.$$

786. (Martyn M. Thomas):—Pairs of tangents are drawn to a closed oval curve, without singularities, at a constant angle 2α; and lines are drawn from their point of intersection, inclined to them, externally, at β and γ. If p and q be the entire lengths of the curves enveloped by these lines, and l that of the given curve, show that

$$p \operatorname{cosec}(\alpha+\beta)=l \operatorname{cosec}\alpha=q \operatorname{cosec}(\alpha+\gamma).$$

787. (M. K. Kewalramani, M. A.):—Prove that if a be not an integer

$$\frac{\pi}{2}\,\frac{f(x+ah)-f(x-ah)}{\sin a\pi}=\frac{f(x+h)-f(x-h)}{1^2-a^2}-2\,\frac{f(x+2h)-f(x-2h)}{2^2-a^2}$$
$$+3\,\frac{f(x+3h)-f(x-3h)}{3^2-a^2}-\dots$$

788. (E. H. Neville):—From the point (u, v) can be drawn four normals to the ellipse $(x^2/a^2)+(y^2/b^2)=1$; four circles are drawn, each through the feet of three of these normals; shew that the sum of the squares of the radii of these circles is given by

$$2a^2b^2\Sigma r^2=(a^2+b^2)^3-(a^2-b^2)(a^2u^2-b^2v^2).$$

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List of Periodicals Received.

(From 16th May to 15th July 1916.)

1. Annals of Mathematics, Vol. 17, No. 3, March 1916.
2. Astrophysical Journal, Vol. 43, Nos. 2 & 3, March and April 1916.
3. Bulletin of the American Mathematical Society, Vol. 22, Nos. 7 & 8, April & May 1916.
4. Bulletin des Sciences Mathematics, Vol. 40, January, February, March & April 9116.
5. L'Intermediaire des Mathematiciens, Vol. 23, Nos. 3 & 4, March & April 1916.
6. Mathematical Gazette, Vol. 8, No. 122, March 1916. (3 Copies).
7. Mathematical Reprints from Educational Times, Vol. 29. (2 Copies).
8. Mathematical Questions and Solutions, Vol. 1, Nos. 5 & 6, May and June 1916. (5 Copies).
9. Mathematics Teacher, Vol. 8, No. 3, March 1916.
10. Messenger of Mathematics, Vol. 45, Nos. 8, 9, 10 & 11. December 1915, January, February and March 1916.
11. Monthly Notices of the Royal Astronomical Society, Vol. 76, Nos. 4, 5, & 6, February, March and April 1916.
12. Philosophical Magazine, Vol. 31, Nos. 185 and 186, May and June 1916.
13. Popular Astronomy, Vol. 24, Nos. 4 & 5, April and May 1916. (3 Copies).
14. Proceedings of the London Mathematical Society, Vol. 15, No. 3, May 1916.
15. Proceedings of the Royal Society of London, Vol. 92, Nos. 640 and 641, April and May 1916.
16. School Science and Mathematics, Vol. 16, Nos. 5 and 6, May and June 1916. (2 Copies).
17. Transactions of the American Mathematical Society, Vol. 17, No. 2, April 1916.
18. Transactions of the Cambridge Philosophical Society, Vol. 22, Nos. 8 & 9, May 1916.
19. Transaction of the Royal Society of London, Vol. 216, Nos. 542, 543 and 544.
20. The Tohoku Mathematical Journal, Vol. 9, Nos. 3, April 1916.
21. Rendiconti Del Circolo Matematico Di Palermo, Vol. 40, No. 2.
22. American Mathematical Monthly, Vol. 23, Nos. 3 and 4, March and April 1916.

The Indian Mathematical Society

(Founded in 1907 for the Advancement of Mathematical Study and Research in India.)

THE JOURNAL

OF THE

Indian Mathematical Society

Vol. VIII.] OCTOBER 1916 [No. 5.

EDITED BY

M. T. NARANIENGAR, M. A.

Hony. Joint Secretary,

WITH THE CO-OPERATION OF

Prof. R. P. PARANJPYE, M.A., B.Sc. Prof. A. C. L. WILKINSON, M.A., F.R.A.S

and others.

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CONTENTS.

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Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. R. KRISHNASWAMI, M.A., 3, Johannes Road, Saidapet, Madras.

Enquiries by intending members and all other communications may be addressed to the Hony. Joint Secretary, D. D. KAPADIA, M.A., B.Sc., 2414, East Street, Poona.

THE JOURNAL

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Indian Mathematical Society.

Vol. VIII.] OCTOBER 1916. [No. 5.

PROGRESS REPORT.

1. The following gentlemen have been elected members at the usual concessional rate :—

1. *Mr. S. Mahadevan, B.A. (Hon.)*—47, Teachers College, Saidapet, Madras ;

2. *Mr. Krishnarao Srinivasarao Karpur, B.A., (Hon.)*—Student, Fergusson College, Poona (City) ;

3. *Mr. C. Bhaskaraiya, B.A. (Hon.)*—Acting Science Demonstrator, Nizam's College ; 5566, Emambowry Bazaar, Secunderabad (Deccan).

2. The following books have been received for the Library—

1. *Exercices Numeriques et Graphiques de Mathematique*—by L. Zoretti, Gauthier-Villars, Paris, 1914 : 7 frs. ;

2. *Test Questions in Junior Algebra*—by F. Rosenberg, University Tutorial Press, London 1916 : 1*s.*/—;

3. *Preliminary Geometry*—by F. Rosenberg, University Tutorial Press, London 1916 : 2*s.*/—;

4. *Descriptive Geometry*—by Dr. G. C. Anthony and Mr. G. F. Ashley (Technical Drawing Series), pub. by D. C. Heath & Co., Bosten, Rs. 3-8 ; Purchased ;

5. *Bombay University Calendar, Part I*, 1916—17.

6. *Calcutta University Calendar, Part III* (Examination Papers), 1916.

3. The Committee have pleasure to announce to the General Body that it has been arranged to hold a *Meeting* of the Society at *Madras* on the 26th, 27th and 28th of December 1916—(the dates being liable to alteration). A local working committee consisting of the following gentlemen has been appointed to take the necessary steps in the matter:—Dewan Bahadur R. Ramachandra Rao, Messrs. E. B. Ross, M. T. Naraniengar, S. Narayana Aiyar, P. V. Seshu Aiyar, P. R. Krishnaswami and C. N. Ganapathi; Mr. P. V. Seshu Aiyar will be the *Local Secretary*.

The Committee expect all the members to be present on the occasion and it is requested that the papers to be read at the meeting will be forwarded in time to either of the Joint Hon. Secretaries, or the Local Secretary.

POONA,
8th Sept. 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

The Legendre Expansion of $F\{(1-2r\cos\theta+r^2)^n\}$.

By M. T. NARANIENGAR.

1. In the March 1916 number of the *Quarterly Journal of Mathematics* * Mr. S. Chapman of Trinity College, Cambridge, investigates a power series for the coefficient of $P_m(\cos\theta)$ in the expansion of $(1-2r\cos\theta+r^2)^n$ in Legendre's functions, and discusses the validity of the result in accordance with Cesáro's method of summability †. The formal analysis set out by Mr. Chapman in § 2 leads to the following expression:

$$(1-2rx+r^2)^n=\Sigma[(2m+1).A_m(r).P_m(x)] \quad \ldots \quad (1)$$

$$\text{where } A_m(r)=(-1)^m r^{-m}\sum_{t=m}\left[{}_nC_t\frac{t_m\cdot(n+\frac{1}{2})_{t-m}}{(t+\frac{1}{2})_t}.\ r^{2t}\right], \quad \ldots \quad (2)$$

and x denotes $\cos\theta$.

In the course of his analysis, which is somewhat tedious, Mr. Chapman uses the formulae for x^t in terms of Legendre's functions given in Todhunter's *Functions of Laplace, Lame and Bessel* (pp. 18, 19), *viz.*

$$x^{2t}=\frac{1}{2t+1}P_0+\frac{2t}{(2t+1)(2t+3)}.5\,P_2+\frac{2t(2t-2)}{(2t+1)(2t+3)(2t+5)}\,9\,P_4+\ldots$$

$$x^{2t+1}=\frac{1}{2t+3}.3\,P_1+\frac{2t}{(2t+3)(2t+5)}.7\,P_3$$

$$+\frac{2t(2t-2)}{(2t+3)(2t+5)(2t+7)}.11\,P_5+\ldots$$

Byerly's *Harmonic Functions*, (p. 50) contains a simpler expression, *viz.*

$$x^t=\frac{t\,!}{1\cdot3\cdot5\ldots(2t+1)}\Big[(2t+1)P_t+(2t-3)\frac{2t+1}{2}P_{t-2}$$

$$+(2t-7)\frac{(2t+1)(2t+3)}{2\cdot4}P_{t-4}+\ldots\Big] \ldots \quad (3)$$

with aid of which Mr. Chapman's result can be more readily established.

2. To this end we shall first determine the co-efficient of r^m in the expansion of $(1-2rx+r^2)^n$ as a power series in x.

We have

$$\begin{aligned}(1-2rx+r^2)^n&=[1+r(r-2x)]^n,\\ &=\sum\left[\frac{n_p}{p!}\,r^p(r-2x)^p\right],\\ &=\Sigma(B_m.r^m),\text{ say.}\end{aligned}$$

---

* *On the Expansion of* $(1-2r\cos\theta+r^2)^n$ *in a series of Legendre's Functions* By S. Chapman, (pp. 16—26.)

† *Cf.* Bromwich: *Infinite Series*, p. 310.

Collecting the terms r^m in the double series, we get

$$B_m = \Big[(-2x)^m \cdot \frac{n_m}{m!} + (-2x)^{m-2} \cdot \frac{(m-1)_1}{1!} \frac{n_{m-1}}{(m-1)!} + (-2x)^{m-4} \frac{(m-2)_2 \cdot n_{m-2}}{2!\,(m-2)!} + \ldots\ldots\Big] \quad \ldots \ (4)$$

The last term in (4) is $\dfrac{n_{\frac{1}{2}m}}{(\frac{1}{2}m)!}$,

if m is even ; and

$$(-2x)^{\frac{1}{2}(m+1)} \cdot \frac{n_{\frac{1}{2}(m+1)}}{[\frac{1}{2}(m+1)]!}$$

if m is odd.

3. Now, to obtain the required Legendre expansion we have to substitute for the several powers of x in $\Sigma\,(B_m r^m)$ from formula (3) and pick out the co-efficients of P_m. For this purpose, we need not consider B_p for values of $p<m$, as such B's will not involve P_m. The expansions of B_{m+1}, B_{m+3}...will likewise be free from P_m. The coefficients of P_m will thus be found from the development of

$$B_m r^m + B_{m+2} r^{m+2} + B_{m+4} r^{m+4} + \ldots$$

and may be written

$$(-2)^m \cdot r^m a_m \cdot \frac{n_m}{m!} + r^{m+2}\Big[(-2)^{m+2} a_{m+2} \frac{n_{m+2}}{(m+2)!} + (-2)^m a_m \frac{(m+1)_1 \cdot n_{m+1}}{1!\,(m+1)!}\Big]$$
$$+ r^{m+4}\Big[(-2)^{m+4} \cdot a_{m+4} \frac{n_{m+4}}{(m+4)!} + (-2)^{m+2} \cdot a_{m+2} \cdot \frac{(m+3)_1}{1!} \frac{(n_{m+3})}{(m+3)!} + (-2)^m \cdot a_m \frac{(m+2)_2}{2!} \frac{n_{m+2}}{(m+2)!}\Big] + \ldots$$

where a_m, a_{m+2}, a_{m+4}...denote the coefficients of P_m in the expansions of x^m, x^{m+2}, x^{m+4},...by (3); that is

$$a_m = \frac{m!}{1.3.5\ldots(2m+1)}(2m+1)$$

$$a_{m+2} = \frac{(m+2)!}{1.3.5\ldots(2m+5)} \cdot (2m+1) \cdot \frac{(2m+5)}{2}$$

$$a_{m+4} = \frac{(m+4)!}{1.3.5\ldots(2m+9)} \cdot (2m+1)\ \frac{(2m+9)(2m+7)}{2.4}.$$

... ...

Comparing with (1) we deduce that

$$A_m(r) = \frac{(-2)^m r^m}{1.3.5\ldots(2m+1)}\Big[\frac{n_m}{m!} + \Big(\frac{2.n_{m+2}}{(2m+3)} + \frac{(m+1)_1}{1!}\ \frac{n_{m+1}}{m+1}\Big) r^2$$

$$+\left(\frac{2^2.n_{m+4}}{(2m+3)(2m+5).2\,!}+\frac{2.n_{m+3}}{(2m+3)}+\frac{n_{m+2}}{2\,!}\right)r^4+\dots\Big].$$

$$=\frac{(-1)^m r^m.n_m}{(m+\frac{1}{2})_m}\Big[1+r^2\frac{(n-m)(n+\frac{1}{2})}{(m+\frac{3}{2})}+\frac{r^4(n-m)_2.(n+\frac{1}{2})_2}{2\,!\,(m+\frac{3}{2})(m+\frac{5}{2})}$$

$$+\dots+\frac{r^{2t}(n-m)_t(n+\frac{1}{2})_t}{t\,!\,(m+t+\frac{1}{2})_t}+\dots\Big], \quad (5)$$

since by Vandermonde's theorem

$$\frac{(m+\frac{5}{2})_2}{2\,!}+\frac{(m+\frac{5}{2})_1(n-m-2)_1}{1\,!\;1\,!}+\frac{(n-m-2)_2}{2\,!}=\frac{(m+\frac{5}{2}+n-m-2)_2}{2\,!}$$

$$=\frac{(n+\frac{1}{2})_2}{2\,!},$$

and generally

$$\frac{(m+t+\frac{1}{2})_t}{t\,!}+\frac{(m+t+\frac{1}{2})_{t-1}(n-m-t)_1}{(t-1)\,!\;1\,!}+\frac{(m+t+\frac{1}{2})_{t-2}(n-m-t)_2}{(t-2)\,!\;2\,!}$$

$$+\dots\frac{(n-m-t)_t}{t\,!}=\frac{(n+\frac{1}{2})_t}{t\,!}$$

Finally, therefore we may write (5) in the form

$$A_m(r)=(-1)^m r^m.\sum_{t=0}\Big[_t\frac{n_{m+t}.(n+\frac{1}{2})_t}{t\,!\,(m+t+\frac{1}{2})_{m+t}}r^{2t}\Big]$$

which is easily seen to agree with Mr. Chapman's result (2).

4. Next, let us consider the more general form $F(1-2r\,x+r^2)$, where $F(z)$ is capable of expansion by Maclaurin's theorem. In this case, $A_m(r)$ takes the form

$$(-1)^m\Big[\frac{r^m}{(m+\frac{1}{2})_m}\sum\Big(p_m.\frac{F^p(o)}{p\,!}\Big)+\frac{r^{m+2}}{(m+\frac{3}{2})_{m+1}}\sum(p+\tfrac{1}{2}).\frac{p_{m+1}}{p\,!}F^p(o)$$

$$+\frac{r^{m+4}}{(m+\frac{5}{2})_{m+2}}\sum\frac{(p+\frac{1}{2})_2.p_{m+2}}{2\,!\;p\,!}F^p(o)+\dots\Big]$$

where the summation includes all integral values of p. Thus we obtain, after some simple reduction,

$$(-1)^m A_m(r)=\frac{r^m}{(m+\frac{1}{2})_m}F^m(1)+\frac{r^{m+2}}{(m+\frac{3}{2})_{m+1}}\left[(m+\tfrac{3}{2})F^{m+1}(1)+F^{m+2}(1)\right]$$

$$+\frac{r^{m+4}}{(m+\frac{5}{2})_{m+2}}\Big[\frac{(m+\frac{5}{2})_2}{2\,!}F^{m+2}(1)+\frac{(m+\frac{5}{2})_1}{1\,!}F^{m+3}(1)+F^{m+4}(1)\Big]+\dots(6)$$

The above result may also be directly derived from the expansion by Taylor's theorem of $F(1-2\,rx+r^2)$, *viz.*

$$\sum\Big[(-2\,rx)^p.\frac{F^p(1+r^2)}{p\,!}\Big]$$

and developing $F^p(1+r^2)$ in powers of r^2, and using (3) for the Legendre expansion of x^p.

5. Lastly, we shall discuss the most general case of $F(z)$ where z stands for $(1-2rx+r^2)^n$, n being any number whatever.

This case differs from the preceding in having (pn) instead of p throughout the right-hand member; so that

$$(-1)^m A_m = \frac{r^m}{(m+\frac{1}{2})_m}\sum\frac{(pn)_m F^p(o)}{p!}$$
$$+\frac{r^{m+2}}{(m+\frac{3}{2})_{m+1}}\sum(pn+\tfrac{1}{2})\frac{(pn)_{m+1}F^p(o)}{p!}+\ldots \quad \ldots \quad (7)$$

The summations in the above are effected by means of the following formulae from Boole's *Finite Differences*;

$$\phi(p)=\phi(o)+\Delta\phi(o).p_1+\Delta^2\phi(o)\,\frac{p_2}{2!}$$
$$+\Delta^3\phi(o)\frac{p_3}{3!}+\ldots\Delta^m\phi(o)\,\frac{p_m}{m!},\ [\text{p. } 11.] \quad \ldots \quad (8)$$
$$\Delta^q\phi(x)=\phi(x+q)-q\phi(x+q-1)$$
$$+\frac{q_2}{2!}\,\phi(x+q-2)-\ldots(-1)^q\phi(x).\ [\text{p. } 19.] \quad \ldots \quad (9)$$

Thus, if we write $\phi_m(p)=(pn)_m$, $\phi_{m+2}(p)=(pn)_{m+1}\,(pn+\frac{1}{2})$, $\phi_{m+4}\,(p)=(pn)_{m+2}\,(pn+\frac{1}{2})_2$, &c.; and expand these by (8), then

$$(-1^m A_m=\frac{r^m}{(m+\frac{1}{2})_m}\left\{\,\phi(o)\frac{F^p(o)}{p!}+\Lambda\phi\frac{F^p}{p-1\,!}+\frac{\Lambda^2\phi F^p}{2\,!\;p-2\,!}+\ldots\right\}+\ldots$$
$$=\frac{r^m}{(m+\frac{1}{2})_m}\left\{\,\phi\left(1+\frac{D}{1!}+\frac{D^2}{2!}\ldots\right)F+\right.$$
$$\left.\Lambda\phi D\left(1+\frac{D}{1!}+\frac{D^2}{2!}+\right)F+\frac{\Lambda^2\phi D^2}{2!}(1+\ldots)F+\right\}+\ldots$$
$$=\frac{r^m}{(m+\frac{1}{2})_m}.e^{\Lambda D}e^{D}[\phi\,F]+\ldots$$
$$=\frac{r^m}{(m+\frac{1}{2})_m}e^{D(1+\Lambda)}\,\phi.F+\ldots$$
$$=e^{DD'}\left\{\frac{r^m\phi_m}{(m+\frac{1}{2})_m}+\frac{r^{m+2}\phi_{m+2}}{(m+\frac{3}{2})_{m+1}}+\ldots\right\}F$$

where D operates on ϕ and D′ on F, and the independent variable is to be put equal to zero after the operations.

Some Deceased Modern Mathematicians.*

1. Weierstrass:—Weierstrass was born in Ostenfelde, Germany, on October 31, 1815, and died at Berlin on February 19, 1897. During the years 1834-38 he studied law and finance at the University of Bonn, and later during 1838-40 he studied mathematics privately under Gudermann at Münster.

Many mathematicians of the first rank gave evidence of their unusual mathematical abilities at an early age. Galois, Gauss, and Abel are instances of this kind. On the other hand there are those who abandoned other lines of work to turn to mathematics, acquiring their special aptitudes along this line at a later period. Weierstrass belongs to the latter class, as he turned to mathematics at an age which exceeded that reached by Galois (1811-1832) and at which Gauss (1777-1855) had already completed his monumental work called *Disquisitiones Arithmeticae.*

Weierstrass constitutes an exception to the supposed rule that elementary teaching is distasteful to those who are specially gifted as regards research ability. He spent a number of years as a teacher of secondary mathematics and began his work as Instructor in the University of Berlin in 1856 in connection with a Technical School.

It was not until 1864 that he received a full Professorship at the University of Berlin and could thus devote all his time to advanced mathematics. He continued to look on his elementary teaching experience with pleasure and had little sympathy with those young men who sought to avoid such experience.

Weirstrass published little compared with the large number of his new results and new methods. He endeared himself to his students by the great liberality with which he allowed them to develop important theories which he himself had started but could not find time to finish. As an instance of this kind we may cite the Doctor's dissertation of Sophie Kowalewski (1850-91) who was one of the most noted women among mathematicians of her time. In her interesting biography of Sophie Kowalewski, Anna Charlotte Leffler remarks that the entire scientific work of Sophie Kowalewski was only the development of the ideas of her great teacher Weierstrass. While this tribute may be too favourable as regards the work of Weirstrass in this connection, it is an evidence of the high regard entertained by his students towards their teacher.

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* Extracted from G. A. Miller's *Historical Introduction to Mathematical Literature*, New York, 1916.

While Weierstrass worked in many different fields of mathematics his work in the Theory of Functions is probably the best known. His example of a continuous function without a derivative, or a continuous curve without a tangent, made a profound impression when it first became known and constitutes one of the most striking results in the history of mathematics.

He recognized early the need of greater rigor in mathematics and is regarded as the greatest exponent of the modern tendency termed by F. Klein "Arithmetizing of Mathematics."†

His theory relating to the Irrational Numbers is of fundamental importance in the modern theory of these numbers. He gave an abstract definition of determinants of order n as a function of n^2 independent variables satisfying three characterestic conditions, which is frequently erroneously attributed to Kronecker (1823-1891). This error occurs, for instance, in the Encyclopedie des Sciences Mathematiques, tome 1, volume 1, p. 90 ‡ [pp. 255-57.]

2. Cayley :—The biography of Cayley is especially interesting to English readers in view of the fact that Cayley comes next to Newton among noted English mathematicians. He is, however, first among the English mathematicians who have advanced modern mathematical theories. Like Vieta, Fermat, Sylvester and Weierstrass, Cayley studied law in his early years.

During the fourteen years which he spent in the practice of law, he wrote between two and three hundred mathematical papers, and these included some of his most important discoveries. He regarded his legal occupations as the means of providing a livelihood, and he reserved with particular care a portion of his time for mathematical, investigations, refusing a considerable part of the legal work which came to him.

Arthur Cayley was born at Richmond, England, on August 16 1821, and died at Cambridge on January 26, 1895. He was educated at Trinity College, Cambridge, and in 1863 he was called to the newly established Sadlerian Professorship of Pure Mathamatics in the University of Cambridge, which position he held until his death. The duty of the incumbent of this position was "to explain and teach the principles of pure mathamatics and to apply himself to the advancement of that science". Few men were better qualified than Cayley for such a position, and few men have carried out more completely the duties implied in the acceptance of the position.

† *Bulletin of the American Mathematical Society*, Vol. 2, (1896), p. 241.

‡ *Cf.* G. Frobenius, Crelle, Vol. 129, (1905) p. 179.

He wrote only one book on mathematics, viz, a *Treatise on Elliptic Functions*, which was published in 1876, and contains a considerable amount of new matter. On the other hand he wrote an unusually large number of papers on a great variety of different topics, including the subjects of quantics, geometry, theory of matrices, symmetric functions, theory of invariants, &c. The number of these papers is almost a thousand and they have been collected and published in thirteen large volumes. The subject with which Cayley's name is perhaps most intimately associated is the theory of algebraic invariants, but he contributed his mite to almost every subject in pure mathematics. In particular, he is generally credited with the introduction of the Absolute into geometry.

Cayley endeared himself especially to Americans by the fact that he lectured for a time at John Hopkins University during its early years, when Sylvester (1814-97) was engaged in the fundamental work of establishing research in this country. Cayley and Sylvester were students at Cambridge at the same time and formed then a lifelong friendship which was doubtless increased by their interest in common subjects of research. In the theory of algebraic invariants they were both among the earliest to make important contributions and during the fifties while Cayley was practising law and Sylvester was an actuary in London, they were in the habit of walking around the courts of Lincoln's Inn discussing the theory of invariants and covariants which was then occupying the attention of both; although both were engaged in other work for a livelihood.

While Cayley and Sylvester would doubtless have become great mathematicians under almost any circumstances, the time at which they lived seemed especially ripe for great advances by English mathematicians. The dispute between Newton and Leibnitz had resulted in an isolation of English mathematicians, since these adopted the less convenient notation of Newton as well as his conservative geometrical methods and thus failed to join directly in many of the active advances which were being made on the Continent. The disadvantages of this isolation were fully realized at the time when Cayley and Sylvester began their scientific work and the general appreciation of the analytical methods developed on the Continent acted as a healthy stimulus for the exercise of their special gifts along this line. [pp. 257-59.]

3. Cremona :—The word Cremona is met with frequently in the form of an adjective in mathematical literature. We speak of a Cremona congruence, a Cremona curve, a Cremona group, a

Cremona substitution, a Cremona transformation, etc. The student of projective geometry is likely to have received inspiration and pleasure from the reading of *Elements of Projective Geometry* which has been translated into various languages.

Luigi Cremona was born in Pavia on December 7, 1830 and died in Rome on June 10, 1903. He was elected as Professor of Geometry in the University of Bologna in 1860 and many of his brilliant discoveries were made soon after. In 1873 he was appointed Professor of Higher Geometry in the University of Rome and Director of the reorganized Engineering School. Most of his important publications preceded this appointment, as the administrative and political duties connected therewith seemed to have consumed most of his energies during his later years.

Cremona was fully identified with Italian institutions and can be called an Italian Mathematician without reserve, unlike Lagrange (1736-1813) who did a greater part of his scientific work in Germany and France. The wonderful Mathematical advances made by Italy since the middle of the nineteenth century were largely guided by Cremona, Brioschi and Beltrami. [pp. 264—5].

4. **Lie** :—The importance of the group-concept in algebraic work was brought to the attention of Mathematicians through the work of Abel and Galois during the first half of the nineteenth century. The fertility of this concept in other lines of work was made clear in the early part of the second half of the century by the researches of Jordan and others. It was not, however, until Lie and Klein had selected this concept as the centralizing and unifying element of their work, that the wide applications of the group concept became generally known and the value of the new method came to be widely appreciated.

Lie permitted his whole soul to be permeated with the group concept and invigourated by its influence he devoted his life with perseverance and great effectiveness to an exposition of various far-reaching Mathematical theories from this new point of view. "The groups do every thing" was the somewhat exaggerated yet inspiring maxim by which he inflamed himself and his pupils.

M. S. Lie was born on December 17, 1842 in Nordfjordeide, Norway and died in Christiania on February 18, 1899. For six and a half years beginning with 1859 he was a student at the University of Christiania, but he did not become specially interested in mathematics. In fact,

after leaving the University he was thinking of devoting himself to Astronomy. A few years later he applied himself to the private study of mathematics and read the classic works of Duhamel, Lame', Chasles, Monge, and Poncelet. He became deeply interested and soon began to make original contributions.

In 1861 he went to Berlin having received a stipend from the University of Christiania as a result of his successful investigations. Here he met Klein and both of them went to Paris later coming under the influence of Jordan Darboux and others. The common scientific interests of Klein and Lie led to several papers which they published jointly. While Lie was in Paris, he made in July 1870 that remarkable discovery of a contact transformation by which a sphere can be made to correspond to a straight line.

In 1872 Lie become Professor in the University of Christiania. This special position did not imply that he was expected to give lectures in the University, but permitted him to devote all his time, undisturbed by teaching duties, to his ivestigations. In 1873 he made the beginning of his extensive theory of transformation groups and in 1873-74 he determined all the finite point and contact transformations of the plane.

In 1876 he helped to organize a new journal of mathematics entitled *Archiv for Mathematik og Naturvidenskab*, and ten years later he accepted a call to assume the duties of a Professorship in the University of Leipzig as a successor to F. Klein, who had accepted a call to go to Gottingen. It was at Leipzig that Lie came in contact with a number of students whom he started in his own work. Among these students there were a number of foreigners who were attracted by his great reputation. We referred above to Lie's absorbing interest in group theory, but it should not be inferred that this theory in its abstract form was his chief interest. It served often as a guiding principle where the development of other theories was his main objective. He was especially interested in the theory of differential equations and regarded this theory as the most important among all mathematical subjects.

To make progress in this theory was his chief aim from the beginning to the end of his productive career. Both his geometrical developments and his theory of continuous groups were subsidiary to this end. His great fame led the Norwegian Government to make

special efforts to secure his return to his own country, and shortly before his death he did return to accept a very honourable position created for him in the University of Christiania. [pp. 265—68.]

5. **Poincare** :—Notwithstanding the rapid increase in the number of prominent mathematicians during the latter half of the nineteenth and the beginning of the twentieth century, and the tendency to withhold full scientific recognition until after the death of an author, Poincare' stood at the beginning of the twentieth century as the one man whom eminent scholars did not hesitate to speak publicly as the greatest living mathematician. Both in pure and in applied mathematics he worked with remarkable success, and during the latter part of his life he devoted considerable attention to philosophical questions.

By his popular treatment of such fundamental questions as the foundations of geometry and the value of science, he did much to spread scientific knowledge and to popularize our science whose beauties are too apt to escape the attention of the world. These beauties were emphasized in his philosophical work.

Poincaré was born at Nancy, France, on April 29, 1854, and died at Paris in July 1912. He was educated successively at the Lycée de Nancy, l'Ecole Polytechnique , and at l'Ecole Nationale Superieure des Mines, receiving his doctors degree from the University of Paris in 1879. He was a very bright student and received first rank at the entrance examination of l'Ecole Polytechnique. At the early age of 32 he was elected as a member of l'Académie des Sciences, and in view of this occasion he prepared in 1884 a statement entitled " Notice sur les travaux scientifiques de M. Henri Poincaré."

Although this Notice was written less than five years after he began the publication of his researches, it reviews a large number of his published articles along the following three lines (1) Differential Equations, (2) General Theory of Functions, (3) Arithmetic or Theory of Numbers. He emphasizes the fact that he did not pursue his researches in these three directions independently of each other, but that the results obtained along these various lines threw light on each other, and that his work along each one of them was greatly aided by the work along the other lines.

The breadth of scholarship exhibited by Poincare' in his early writings and his great ability to observe relations between apparently widely different subjects became still more pronounced as he grew older,

but we observe even at this early date a mind of very broad sympathies and of extraordinary ability to generalize. His principal writings may be classed under the following four headings :—pure mathamatics, analytic and celestial mechanics, mathematical physics, and the philosophy of science.

In 1909 Emile Borel published in the Journal called *La Revue du Mois*, an article on the method of Poincarè. Parts of this were translated for *the Bulletin of the Calcutta Mathematical Society*. We quote from the translation :—

The method of Poincare′ is essentially active and constructive. He approaches a question, acquaints himself with its present conditions without being much concerned about its history, finds out immediately the new analytical formulas by which the question can be advanced, deduces hastily the essential results, and then passes on to another question. After having finished the writing cf a memoir, he is sure to pause for a while, and to think out how the exposition could be improved ; but he would not, for a single instance, indulge in the idea of devoting several days to didactic work. Those days could be better utilized in exploring new regions.

"All this is not specially applicable to mathematics. Let us examine more closely the mechanism made use of for discovery. The essential feature of that mechanism is, as we have already pointed out, the construction of new forumlas. It is not useless that some stress is laid on this point ; for this constructive power is the essential trait of the geinus of Poincare′. The non-mathematical readers can be made to understand all this by means of a comparison. They know what arithmetical calculation is, and are often led to believe that mathematicians are in the habit of making interminable additions, multiplications, etc., and also extractions of cube roots.

In reality arithmetical operations are unique combinations of integral numbers formed of units which are equal to one another. These operations can be compared to the construction of regular walls by means of bricks of uniform size. The work requires only some patience and a little care. On the contrary, analytical operations make use to extremely numerous materials, and their variety is comparable to those of structures where stone, marble, wood, iron, etc, are used. These operations are as different from each other as cuirasse is from a Gothic church. They have also, with the architectural construction, this in common that an impression of beauty is produced by

the simplicity and elegance of the essential lines, without exhibiting any of the efforts by means of which the result has been obtained."

Poincaré was a great pioneer boldly entering into unexplored regions and noting some of the most important objective points, and then leaving to others the details of organization. In the words of Borel 'he was more of a conqueror than a colonizer', and he attached little importance to conceptions which cannot be realized in a concrete form. In this respect he may be compared with men of action; his method of work was too active to have much room for such reflections as do not lead to concrete results.

Poincaré won great fame in connection with his prize memoir relating to the problem of three bodies. In 1885 King Oscar II of Sweden offered a prize for the solution of a question in reference to this general problem, and one half of this prize was awarded to Poincare for his article entitled "Sur le proble'me des trois corps et les e'quations de la dynamique" published in the *Acta Mathematica* in 1890. In the *Bibliotheca Mathematica* for 1904 page 198, Eneström calls attention to the interesting fact that the copy of this memoir for which the prize had been actually awarded contained a serious error, and that the given published article was really prepared for the press after the prize had been awarded.

To those who would like to see a connection established between the university athlete and the intellectual giant, between physical powers and intellectual greatness, Poincaré was a decided disappointment. He was only 5 feet 5 inches in height, was somewhat stooped—at least in the latter part of his life, and weighed 154 pounds. Even as a child he was rather weak and did not engage in the rougher sports of the boys of his age. He cared little for politics and achieved his greatness solely through his scholarly services. When he entered the French Academy he was told that he was born a mathematician. He had, however, the good fortune to live in a country where mathematical attainments are held in high esteem even by the general public.

He wrote a number of books especially on mathematical physics, but the three books most commonly known deal with philosophical questions and bear the following titles: *La Science et l'Hypothese, La Valeur la Science*, and *Science et Methode*. The first had a circulation of 16000 copies, and had increased his personality tenfold.

The great mainspring of Poincarés activity was seeking the truth. This made his life both simple and beautiful. Seeking the truth implies

an open acknowledgment of ignorance. In fact, one of the strongest mathematical methods consists in putting an x for the unknown quantity ; but how could we do this unless we were willing to admit our ignorance of the value of the unknown. Even in his mature years Ponicare' could honestly ask the question "La terre tourne-t-elle ?" Things that are commonly accepted as true, but have not been fully established, frequently offer the most important fields of research, and the great investigator does not always accept the views of the masses as evidence of truth.

At the funeral of Poincaré, the French Minister of Public Instruction remarked that all his work, all his life, was animated, by a prepossession which found expressed in this thought : "The search for truth must be the goal of our activity ; it is the only end that would be worthy of it."

[pp. 268-74.]

SHORT NOTES.

A further note on Question 567.

[The solution given in the "Messenger of Mathematics" and on p. 102 of this Journal is unsymmetrical. The following application of the same method preserves symmetry and leads directly to the actual system of equimomental particles required.]

Let each *face* of a tetrahedron be given a surface-density which is uniform, and let the ratios of the densities be such that the faces have all the same mass; since a uniform triangle of mass m is equimomental with three particles each of mass $m/3$ at the mid-points of the sides, the *hollow* tetrahedron so described is equimomental with six particles of equal masses at the mid-points of its edges. It follows at once that if P, Q, R, U, V, W are the mid-points of the edges of a uniform *solid* tetrahedron of mass M, the tetrahedron is equimomental with a distribution along the three lines PU, QV, RW which meet in the centroid G, the total mass of each of the three lines being M/3 and the density in each line bring proportional to the square of distance from G, which is the centroid not only of the tetrahedron but also of each line. With a distribution of this kind in a line, the ratio of the square of the radius of gyration about G to the square of the half-length is

$$\int_0^1 t^4\,dt \Big/ \int_0^1 t^2\,dt,$$

that is, 3/5, and therefore, if the mass of the line is n, the line is equimomental with the system obtained by placing particles of mass $3n/10$ at each end and the residue at the centre. Hence the solid tetrahedron is equimomental with the system obtained by placing particles of mass M/10 at the mid-points of the edges and the residue at the centroid.

The reader is recommended to apply the corresponding process to the triangle; he will obtain at once a system much simpler than that of thirteen particles given inthe June number of the Journal.

ERIC H. NEVILLE.

Infinite Power Chains.

In his Note on the '*Convergence of Infinite Power Chains*', published in the J. I. M. S., Vol. VIII, page 11, Mr. A. Narasinga Rao, B.A. has taken the initial term of the sequence to be unity. The object of the present note is to examine the range within which the initial term a_1 should lie to preserve the convergence of the sequence considered.

The sequence (a_n), where $a_{n+1}=a^{a_n}$ converges, if at all, to a root of the equation $x=a^x$, *i.e.* a point of intersection of the straight line $y=x$ and the exponential curve $y=a^x$.

The following results were arrived at by Mr. A. Narasinga Rao :

(i) $e^{e^{-1}}<a$,	no real root,	convergence impossible ;
(ii) $e^{e^{-1}}=a$,	one real root,	convergence possible ;
(iii) $1<a<e^{e^{-1}}$,	two real roots,	convergence possible ;
(iv) $0<a<1$,	one real root,	convergent, though oscillating.

(i) The first case requires no further consideration.

(ii) When $a=e^{e^{-1}}$, the limit to which the sequence may converge is e, and there are three special cases to be considered.

First. If $e<a_1$, the sequence diverges to ∞.

For, in this case, the sequence is an increasing monotone; and given any number N, however great, we can find a term in the sequence which is greater than N.

The graphical discussion of this and the succeeding cases is interesting. Let us represent the sequence (see Fig. I.) by points on the line $y=x$, the point D_n, whose abscissa is a_n, representing a_n. Then, the construction for D_{n+1} is as follows. Let the ordinate through D_n cut the exponential curve in D'_{n+1}. Let the line through D'_{n+1} parallel to the x-axis cut the line $y=x$ at D_{n+1}; then D_{n+1} shall represent a_{n+1}. By means of this construction, we get the range of points D_1, D_2...which is readily seen to diverge to ∞.

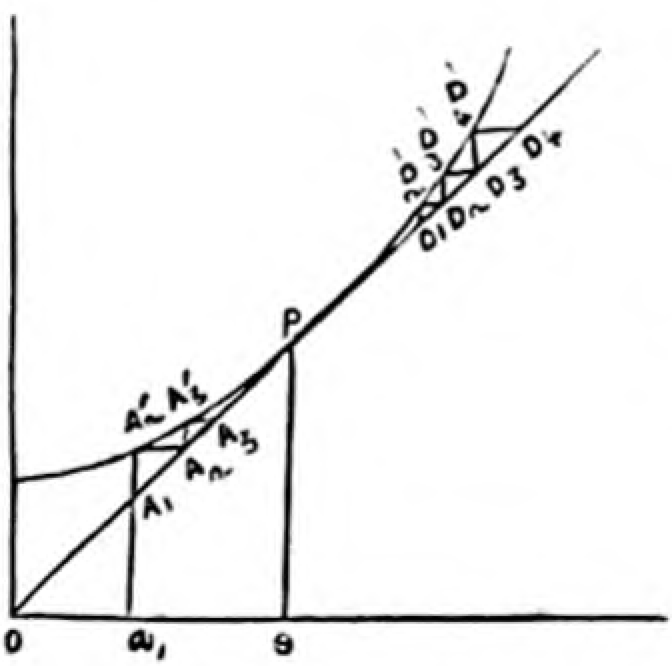


Fig. I.

Secondly. If $a_1=e$, the sequence converges to e.

For, each term of the sequence is e.

Thirdly. If $a_1 < e$ (a_1 should be finite), the sequence converges to e

For, starting with any point A_1, which corresponds to any value of a_1 which is less than e, we get the range of points A_1, A_2...which ascend up the straight line $y=x$, but cannot cross the point P $(x=e)$ and hence converge to e.

(iii) When $1 < a < e^{e^{-1}}$, the two possible values to which the sequence may converge are x_1 and x_2, $(x_1 < x_2)$ the two roots of the equation $x = a^x$.

Here, we have five special cases to consider.

(a) If $x_2 < a_1$, the sequence diverges to ∞.

(b) If $x_2 = a_1$, the sequence converges to x_2.

These results follows readily for reasons similar to those given in (ii).

(c) If $x_1 < a_1 < x_2$, the sequence converges to x_1.

For, in this case, the sequence is a decreasing monotone, none of whose terms is less than x_1. Graphically, (see Fig. II) we have the range of points A_1, A_2...which descend down the straight line $y=x$ and converge to the point P_1.

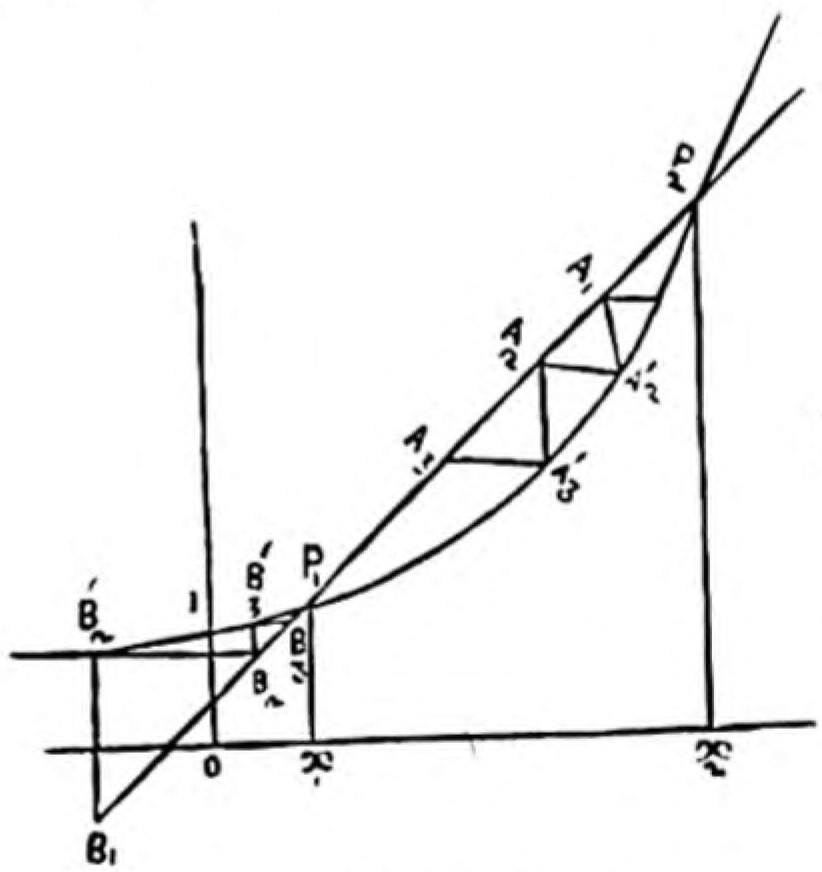


Fig. II.

(d) If $x_1 = a_1$, the sequence converges to x_1.

(e) If $a_1 < x_1$ (a_1 should be finite), the sequence converges to x_1.

For, in this case, the sequence is an increasing monotone none of whose terms can be greater than x_1. Graphically, (see Fig. II.) we have the range of points B_1, B_2...which ascend up the straight line $y=x$ and converge to P_1.

(iv) If $0 < a < 1$, the sequence oscillates and can be split up into two parts, consisting of the odd terms and the even terms respectively, each part being monotonic. If x_1 is the root of $x = a^x$, in this case, and

if $a_1 < x_1$, the sequence of odd terms is an increasing monotone and the sequence of even terms is a decreasing one. And if $x_1 < a_1$, the former is decreasing and the latter is increasing. In either case, both the sequences converge to the same limit x_1. Graphically, we have the range of points A_1, A_3...on one side of P_1 and the range A_2, A_4...on the other ide, each converging to P_1. (See Fig. III.)

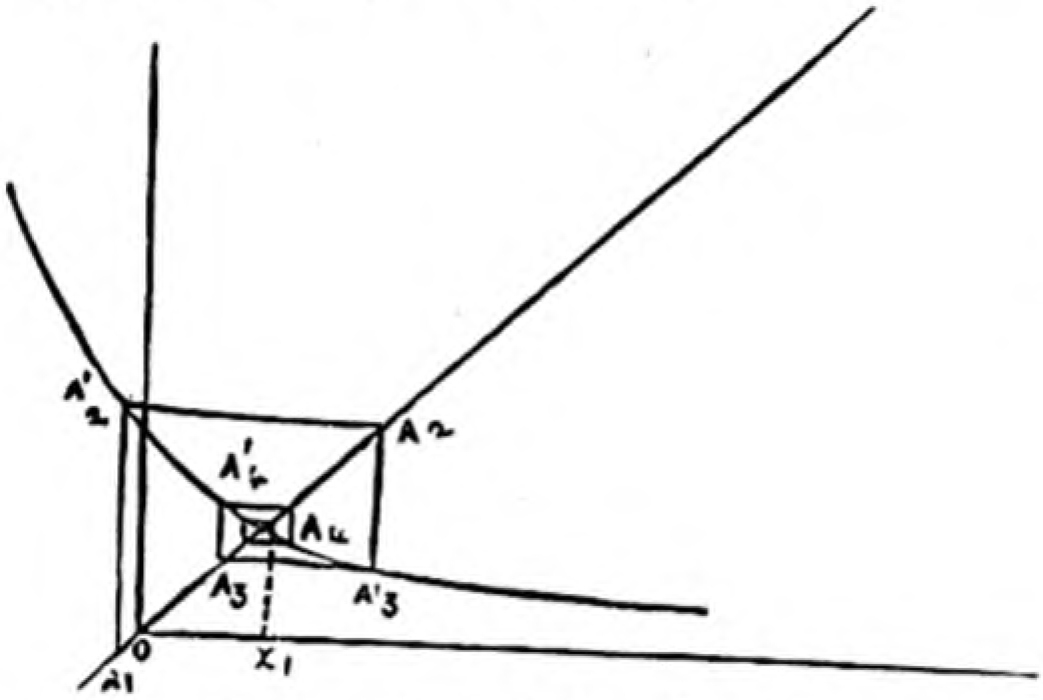


Fig. III.

The different results may be tabulated as follows :—

a	a_1	Limit.
$e^{e^{-1}} < a$	a_1 (finite).	∞
$e^{e^{-1}} = a$	$e < a_1$	∞
	$a_1 \leq e$	e
$1 < a < e^{e^{-1}}$	$x_2 < a_1$	∞
	$x_2 = a_1$	x_2 (the greater root).
	$a_1 < x_2$	x_1 (the smaller root).
$o < a < 1$	a_1 (finite).	x_1 (the only root).

S. R. RANGANATHAN.

Note on Question 742.

[Q. 742. (MARTYN M. THOMAS):—Shew that a curve and all its pedals, positive and negative, have the same potential at the pedal origin.]

The following general theorem may be stated:

Let $x_0=x_0\ (t_0)$, $y_0=y_0\ (t_0)$ be the equations of a curve; and let $x_1=x_1\ (t_1)$, $y_1=y_1(t_1)$ be those of an associated curve, such that

$$t_0-t_1+\lambda=z=f(x_0,\ y_0,\ dy_0/dx_0,\ldots)$$
$$=f(x_1,\ y_1,\ dy_1/dx_1,\ldots);$$

$=$a quantity related to the curves $(x_0\,y_0)$. $(x_1\,y_1)$ geometrically in the same manner.

Then

$$\int_0^{t_0}\phi'(z)dt_0=\int_0^{t_1}\phi'(z)dt_1+\int_0^{z}\phi'(z)dz$$
$$=\int_0^{t_1}\phi'(z)dt_1+[\phi(z)]$$

provided $\phi(z)$ is a single valued function.

Cor. In the case of a closed curve C_0

$$\int_{C_0}\phi'(z)dt_0=\int_{C_1}\phi'(z)d_1$$

since $\int\phi'(z)dz$ round a closed curve is zero.

Applications:

1. The potential of a closed curve is equal to that of its n^{th} positive or negative pedal.

For $V_0=\int_{C_0}\frac{ds}{r}=\int_{C_0}\frac{d\theta_0}{\sin\phi_0}=\int\frac{d\theta_1}{\sin\phi_1}$, $\left[\begin{array}{l}\theta_0=\theta_1+\frac{\pi}{2}\phi_0\\ \theta_0=\phi_1\end{array}\right]$

since $\int\frac{d\phi}{\sin\phi}$ round a closed curve is zero.

Hence $V_0=V_1=V_2=\ldots V_n$, where n is positive or negative.

2. When n is any fraction, the $nth.$ pedal is geometrically defined as in the article on *Root Pedals*, J. I. M. S., Vol. II, p. 99.

In this case also, proceeding as before we have $V_0=V_n$.

3. For a similar reason the potentials of the inverse and the polar reciprocal of a closed curve are the same.

4. In the case of an *arc* of a curve (not closed)

$$V_o = V_n - \int \frac{d\phi}{\sin\phi} = V_n - [\log \tan \tfrac{1}{2}\phi]$$

$$= V_n - \log(\tan \tfrac{1}{2}\beta . \cot \tfrac{1}{2}\alpha)$$

β, α being the limiting values of ϕ at the extremities of the arc.

5. Let $I_o = \int_{C_o} f(\phi)\, d\theta_o$ round a closed curve,

and $\quad I_n = \int_{C_n} f(\phi)\, d\theta_n$, the corresponding integral of the n^{th} pedal; then $I_o = I_n$.

For, $\int f(\phi)\, d\phi = 0$, round a closed curve, provided $f(\phi)$ is a single-valued function of ϕ. Hence the result stated.

M. T. NARANIENGAR.

A small theorem.

By giving particular values to $f(x)$ in the following theorem, neat indentities can be obtained; the theorem is:

$$\frac{f(o)}{a} + \frac{f'(o)}{1!}\frac{x}{a+1} + \frac{f''(o)}{2!}\frac{x^2}{a+2} + \frac{f'''(o)}{3!}\frac{x^3}{a+3} + \ldots$$

$$= \frac{f(x)}{a} - \frac{xf'(x)}{a(a+1)} + \frac{x^2f''(x)}{a(a+1)(a+2)} - \frac{x^3f'''(x)}{a(a+1)(a+2)(a+3)} + \ldots$$

This may be demonstrated as follows :—

$$\int_0^1 z^{a-1} f(zx)dz = \int_0^1 z^{a-1} f\left\{x - x(1-z)\right\} dz.$$

Expanding each side by Taylor's theorem, we have

$$\int_0^1 z^{a-1}\left\{f(o) + zxf'(o) + \frac{z^2x^2}{2!}f''(o) + \ldots\right\} dz$$

$$= \int_0^1 z^{a-1}\left\{f(x) - x(1-z)f'(x) + \frac{x^2(1-z)^2}{2!}f''(x) - \ldots\right\} dz.$$

Integrating each term separately we get the theorem.

S. NARAYANA AIYAR.

The Face of the Sky for November & December 1916.

The Sun

enters Sagittarius on November 22 at 8 P. M. and Capricorns on December 22 at 9 A. M.

Phases of the Moon.

	November.			December.		
	D.	H.	M.	D.	H.	M.
First Quarter ...	4	16	26	3	23	0
Full Moon ...	11	8	31	10	19	1
Last Quarter ...	18	17	35	18	13	8
New Moon ...	26	19	34	26	8	37

Eclipses.

There is a partial eclipse of the Sun on December 25 invisible except in the Antartic regions.

The Planets.

Mercury is in superior conjunction with the Sun on November 23 It is in conjunction with the Moon on November 24 and on December 25, with Mars on December 22, with *m* Virginis on November 1 and with θ Ophinchus on December 5.

Venus is in conjunction with the Moon on November 22 and on December 6. It is in conjunction with θ Virginis on November 16.

Mars is in conjunction with the Moon on November 27 and on December 26.

Jupiter is stationary on December 20. It is in conjunction with the Moon on November 8 at 8-30 P. M. and on December 5 at 10-30 P. M.

Saturn is stationary on November 11. It is in conjunction with the Moon on November 16 and on December 13.

Uranus is in quadrature to the Sun on November 8. It is in conjunction with the Moon on November 3, November 30 and December 28.

Neptune is stationary on November 7. It is in conjunction with the Moon on November 16 and December 13.

V. Ramesam.

SOLUTIONS.

Question 613.

(P. V. SESHU IYER) :—Show that

$$\operatorname*{Lim}_{z\to o} \frac{\dfrac{P(s)}{P(s+z)}\left(\dfrac{x}{2}\right)^{z}-\dfrac{P(n+s)}{P(n+s-z)}\left(\dfrac{x}{2}\right)^{-z}}{z}$$

$$=2\log\frac{x}{2}-\psi(s)-\psi(n+s),$$

where $\psi(x)$ stands for $\dfrac{d\log Px}{dx}$, and P (x) denotes $\Gamma(x+1)$.

Remarks and Solution by K. B. Madhava.

Let us write $f(z)$ for the expression in the numerator.

Then $$\lim_{z\to o}\frac{f(z)}{z}=f'(o).$$

$$\text{Now } f'(z)=\frac{P(s)\left(\dfrac{x}{2}\right)^{z}\log\dfrac{x}{2}}{P(s+z)}-\frac{P(s)\left(\dfrac{x}{2}\right)^{z}P'(s+z)}{[P(s+z)]^2}$$
$$+\frac{P(n+s)\left(\dfrac{x}{2}\right)^{-z}\log\dfrac{x}{2}}{P(n+s-z)}-\frac{P(n+s)\left(\dfrac{x}{2}\right)^{-z}P'(n+s-z)}{[P(n+s-z)]^2}.$$

$\therefore$ $f'(o)=2\log\frac{x}{2}-\psi(s)-\psi(n+s)$, where ψ is as defined

Here z is made to approach zero through positive values; if otherwise it is easy to see that the second and third terms enter with changed signs. The limit that we have just calculated has a very interesting application in Bessel Functions, see for instance, Gray and Mathews.

For let it be required to express

$$x^n\int_0^{\frac{\pi}{2}}\sin(x\sin\theta)\cos^{2n}\theta\,d\theta-x^n\int_0^{\infty}e^{-x\sinh u}\cosh^{2n}u\,du \quad \ldots \quad (A)$$

in terms of Bessel and Neumann functions.

This can of course be done by the usual definite integral definition of these functions, but if we take the integral

$$I=x^n\int_{+\infty i}^{+\infty i}e^{ixz}(1-z^2)^{n-\frac{1}{2}}dz$$

along a contour through the imaginary axis beginning at $+\infty i$ and ending again there, but only enclosing the points -1 and $+1$, we can successively, by Cauchy's theorem, deduce (with some restrictions on the magnitudes of n and z) results to establish the equation

$$G_n\sin n\pi=J_n\cos n\pi-J_{-n} \quad \ldots \quad \ldots \quad (B)$$

where $\sqrt{\pi}2^{n-1}\Gamma(n+\frac{1}{2})\,G_n$ is the expression (A)

When n is an integer however, (B) gives an apparently indeterminate form for G_n; in such a case, we need only understand (B) to mean

$$\lim_{z\to 0} \sin (n-z)\pi.\ G_{n-z}=J_{n-z}\cos(n-z)\pi-J_{-n+z}$$

i.e. $$\pi G_n=\lim_{z\to 0}\frac{(-)^n J_{-(n-z)}-J_{(n-z)}}{z}$$

$$=\lim_{z\to 0}\left\{\left(\frac{x}{2}\right)^{-n+z}\sum_{n=0}^{n-1}\frac{(-)^{n+r}}{\Gamma(r+1)}\cdot\frac{1}{z.\Gamma(-n+r+1+z)}\left(\frac{x}{2}\right)^{z}\right.$$

$$\left.+\left(\frac{x}{2}\right)^{n}\sum_{r=0}^{\infty}(-)^r\frac{F(z)}{z}\left(\frac{x}{z}\right)^{2r}\right\}$$

where $F(z)=\Gamma(r+1)\Gamma(n+r+1)f(z)$. Putting in the limit we have determined

$$\frac{\pi}{2}G^n=Y_n-J_n\,(\log 2+\psi(0))$$

where J, Y are Bessel and Neumann functions, respectively.

Here ψ (0) of Mr. Seshu Iyer's notation

$=\psi$ (1) of Dr. Bromwich's $=-C$. (Euler's constant).

[See Bromwich, P. 475, Ex. 42].

For other properties of Hankel's function G_n reference may be made to §§ 17.6 and 17 61 of Whittaker and Watson "*Modern Analysis*" (Revised Edition, 1915).

Question 652.

(S. P. Singaravelu Mudaliar) :—From a point (eccentric angle ϕ) of an ellipse of semi-axes a, b, three normals are drawn to the ellipse; shew that the square on the radius of the circle passing through the feet of the normals is

$$(a+b^2/2a)\cos^2\phi+(b+a^2/2b)\sin^2\phi$$

Additional solution by E. H. Neville.

We prove first that

If a circle cuts an ellipse in the points whose eccentric angles are θ_1, θ_2, θ_3, θ_4, then the centre of the circle has co-ordinates

$$\{(a^2-b^2)/4a\}\,\Sigma\cos\theta_r,\ \{(b^2-a^2)/4b\}\,\Sigma\sin\theta_r$$

a familiar theorem, virtually given in Wolstenholme's *Mathematical Problems* (No. 1026 of the 3rd Edition). If the radius of the circle is r and the centre is p, q, the four eccentric angles are given by the equation

$$(a\cos\theta-p)^2+(b\sin\theta-q)^2=r^2;$$

arranged as an equation in cos θ, this is

$$\{(a\cos\theta-p)^2+b^2(1-\cos^2\theta)+q^2-r^2\}^2=4\,b^2q^2\,(1-\cos^2\theta)$$

and an inspection of the coefficients of $\cos^4\theta$ and $\cos^3\theta$ gives

$$(a^2-b^2)\,\Sigma\cos\theta_r=4\,ap,$$

and the corresponding equation

$$(b^2-a^2)\,\Sigma\sin\theta_r=4\,bq,$$

follows from symmetry.

We can now solve Q. 652; if ψ_1, ψ_2, ψ_3 are the eccentric angles of the feet of the normals from the point of eccentric angle ϕ, then ψ_1, ψ_2, ψ_3, ϕ are the four angles given by

$$a^2\cos\phi\sec\psi-b^2\sin\phi\operatorname{cosec}\psi=a^2-b^2;$$

arranged as an equation in $\cos\psi$ this is

$$\{(a^2-b^2)\cos\psi-a^2\cos\phi\}^2(1-\cos^2\psi)=b^4\sin^2\phi\cos^2\psi,$$

and therefore

$$(a^2-b^2)\{\cos\phi+\Sigma\cos\psi_r\}=2\,a^2\cos\phi, \qquad \ldots \qquad \ldots \quad (1)$$

from which follows by symmetry

$$(b^2-a^2)\{\sin\phi+\Sigma\sin\psi_r\}=2b^2\sin\phi. \qquad \ldots \qquad \ldots \quad (2)$$

Since the circle through the feet of the normals passes also through the point diametrically opposite to the given point, the centre of the circle has co-ordinates

$$\{(a^2-b^2)/4a\}\{(\Sigma\cos\psi_r)-\cos\phi\},\ \{(b^2-a^2)4b\}\{(\Sigma\sin\psi_r)-\sin\phi\}$$

and by (1), (2) these are equal to $(b^2/2a)\cos\phi$, $(a^2/2b)\sin\phi$. The circle with this centre which passes through the point $(-a\cos\phi, -b\sin\phi)$ has the square of its radius equal to

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

The result which we have to prove, and Q. 652 can be expressed differently:

If α, β, γ are the executive angles of the vertices of a triangle inscribed in an ellipse, and if δ denotes $-(\alpha+\beta+\gamma)$, then the circumcentre has co-ordinates:

$$\{(a^2-b^2)/4a\}(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),$$
$$\{(b^2-a^2)/4b\}(\sin\alpha+\sin\beta+\sin\gamma+\sin\delta).$$

Since the centroid has co-ordinates

$$(a/3)(\cos\alpha+\cos\beta+\cos\gamma),\ (b/3)(\sin\alpha+\sin\beta+\sin\gamma)$$

and is the point of trisection nearer to the circumcentre of the line joining the circumcentre, the first co-ordinate of the ortho-centre is

$$3(a/3)(\cos\alpha+\cos\beta+\cos\gamma)-2\{(a^2-b^2)/4a\}(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),$$

that is,

$$\{(a^2+b^2)/2a\}(\cos\alpha+\cos\beta+\cos\gamma)-\{(a^2-b^2)/2a\}\cos\delta,$$

and the second co-ordinate of the ortho-centre is

$$\{(b^2+a^2)/2b\}(\sin\alpha+\sin\beta+\sin\gamma)-\{b^2-a^2)/2b\}\sin\delta;$$

these expressions also are given by Wolstenholme in the problem quoted.

The above solution of Q. 652 depends on the same equation as that given by Mr. K. B. Madhava on p. 110; the principal difference is that instead of referring to the complete equations we observe only certain coefficients which are sufficient for our purpose; there is a slight difference also in the fact that the form required for the square of the radius is exhibited as natural and not as an artificial transformation.

Question 655.

(A. Narasinga Rao) :—Solve for f from the relation

$$f'''(x)=\alpha.f(x+\beta),$$

α and β being any constants

A curve and its *nth.* evolute are similar. Find the intrinsic equation of the curve.

Solution by N. Durairajan.

In similar polygons the ratios of corresponding sides are equal, and angles at corresponding points are equal. Considering the curves to be limits of polygons of an infinite number of sides, we see that at corresponding points,

$$\text{(i) } ds : ds' = \text{const., (ii) } 180^\circ-\delta\psi=180^\circ-\delta\psi'.$$

$\therefore\ \rho/\rho'=$const.; and $\psi=\psi'+\beta$ (β being a constant).

For inversely similar figures,

$$\rho/\rho'=\text{const.; and } \psi+\psi'=\beta.$$

Let $\rho=f(\psi)$, $\rho=g(\psi)$ be the equations to the two curves. If the point ψ_1 on the first corresponds to the point ψ_2 on the second, then

$$\rho_1/\rho_2=k,\ \psi_1=\psi_2+\beta.$$

$$\therefore\qquad f(\psi_2+\beta)=k.g(\psi_2).$$

The condition of similarity is that the functions f and g should be related in such a manner as to permit of the constants k and β being so chosen as to have $f(\psi_2+\beta)\equiv k.g(\psi_2)$ for all values of ψ_2.

In the case of a curve and its n^{th} evolute: let $\rho=f(\psi)$ be the curve. The first normal being at point 'α,' the radius of curvature of the n^{th} evolute is

$$\left[\left(\frac{d}{d\psi}\right)^n f(\psi)\right]$$

when $\psi=\alpha$.

But the inclination of the tangent there is $\alpha+n\frac{\pi}{2}$.

The equation of the n^{th} evolute is $\rho=f^{(n)}\left(\psi-n\frac{\pi}{2}\right)$

The condition of similarity is

$$k.f^{(n)}\left(\psi-n\frac{\pi}{2}\right)=f\left(\psi-n\frac{\pi}{2}+\beta\right)$$

and for inversely similar curves

$$k f^{(n)}\left(\psi-n\frac{\pi}{2}\right)=f\left(\psi-n\frac{\pi}{2}+\beta\right).$$

Hence we have to solve equations of the type

$$\left(\frac{d}{dx}\right)^n . f(x)=\alpha f(\pm x+\beta) \qquad \dots \quad (1)$$

Writing $y=f(x)$

$$y^{(n)}=\alpha . e^{\beta\frac{d}{dx}} . y$$

i.e.

$$D^n=\alpha e^{\beta D} \text{ where } D=\frac{d}{dx}.$$

This is a transcendental equation and is not in general solvable. One solvable case is got by assuming $\beta=0$, so that

$$D^n=\alpha=a^n \text{ say.}$$

If $n=2m$, the primitive is (Forsyth: *Differential Equations.*)

$$y=Ce^{-ax}+De^{ax}$$

$$+\sum_{r=1}^{m-1} e^{ax\cos\frac{r\pi}{m}}\left[A_r \cos ax \sin\frac{r\pi}{m}+B_r \sin ax \sin\frac{r\pi}{m}\right].$$

Question 680.

(R. VYTHYNATHASWAMY):—If $l, m, n \dots \lambda$ are all positive integers, find the greatest value of

$$lx+my+nz+\dots$$

subject to the condition

$$x+y+z+\dots=\lambda,$$

$x, y, z, \dots$ being zero or positive integers.

Solution by A. Narasinga Rao, D. R. Karve and H. K. Chakraburty, B. Sc.

Let the coefficients arranged in the order of descending magnitude be l, m, n...Then the maxium value is obviously $l\lambda$. For

$$l\lambda-(lx+my+\ldots)$$
$$=l(x+y+z\ldots)-(lx+my\ldots)$$
$$=y(l-m)+z(l-n)+\ldots$$

which is an essentially positive quantity, since $l>m>n,\ldots$

In the same way, we may show that the minimum value of this function is $r\lambda$ where r is the least of the quantities l, m, n...

Question 742.

(Martyn M. Thomas, M.A.) :—Show that a curve and all its pedals, positive and negative, have the same potential, at the pedal origin.

Solution by K. B. Madhava.

Evidently the proposer has in view the law of the inverse square; and with that assumption, consider the potential V at any point O for a thin uniform bar of small cross section k and mean density ρ.

$$\text{Then } V=\gamma k\rho \int\frac{ds}{OP}$$
$$=\gamma k\rho\int_{A}^{\pi-B}\frac{d\theta}{\sin\theta}$$
$$=\gamma k\rho \log \cot\frac{OAB}{2}\cot\frac{OBA}{2}.$$

Now let ABCD...NA be any closed polygon of any number of sides with no reentrant angles, and let A′ B′...N′ A′ be its 'pedal polygon' with respect to the origin O.

Then the total potential V of the polygon ABC...NA

$$=\gamma k\rho \log\cot\frac{OAB}{2}\cot\frac{OBA}{2}$$
$$=\gamma k\rho \log\cot\frac{OBC}{2}\cot\frac{OCB}{2}+\ldots$$
$$=\gamma k\rho \log\left(\cot\frac{OAB}{2}\cot\frac{OBA}{2}\cot\frac{OBC}{2}\cot\frac{OCB}{2}\ldots\right)$$
$$=\gamma k\rho \log\left(\cot\frac{OA'B'}{2}\cot\frac{OB'A'}{2}\cot\frac{OB'C'}{2}\cot\frac{OC'B'}{2}\ldots\right)$$

$=$V of the 'pedal polygon.' (Cf. Minchin p. 310.)

Indefinitely increasing the number of sides of the polygon which in the limit becomes a curve, we have the theorem stated in the problem for the positive pedals. Considering any of these positive pedals as the original curve we see how to extend the theorem for all pedals, positive or negative.

Question 691.

(A. Narasinga Rao) :—A marble slab of n pounds breaks into k pieces with which a tradesman finds himself able to weigh goods from 1 to n pounds (fractions excluded). Show that the least value of k is the smallest integer satisfying the relation $3^k \geq 2n+1$.

What are the weights of the several pieces?

Solution by N. Durairajan.

Let $a_1\ a_2 \ldots a_k$ be the k pieces, so that $a_1+a_2+\ldots a_3=n$. Also the weights may be put simultaneously in both scales if necessary. Now consider the product $P\equiv(1+x^{a_1}+x^{-a_1})(1+x_a{}^2+x^{-a_2})\ldots(1+x^{ak}+x^{-ak})$

(1) This product has coefficient of x^r=coefficient of x^{-r}, as is seen by changing x into x^{-1};

(2) The indices represent all possible combinations of $a_1 \ldots a_k$, either additively or subtractively;

(3) The coefficients of all terms are positive integers $\geq$ unity.

Hence $P\equiv 1+\Sigma p_r(x^r+x^{-r})$ ... ... (1)

Now, if $a_1 \ldots a_k$ be the k pieces with which we can weigh out 1 to n lbs., r in (1) has all values from 1 to n.

Hence
$$P\equiv 1+\sum_1^n p_r(x^r+x^{-r}).$$

When $x=1$, left side$=3^k$; and since $p_r \geq$ an right side is $\geq 1+2n$.

Thus
$$3^k \geq 2n+1.$$

This is a necessary condition. In the critical case $3^k=2n+1$, $p_1=p_2=\ldots p_n$; and

$$P=\frac{(1+x^{a_1}+x^{2a_1})\ldots}{x^{a_1+\ldots a_k}}=\frac{1+x+\ldots x^{2n}}{x^n}.$$

$\therefore\ P\,(1+x^{a_1}+x^{2a_1})=1+x+\ldots x^{3^k-1}=(1+x+x^2)(1+x^3+x^6)\ldots$

Thus, if a_1 be the least of the k quantities $a_1 \ldots a_k$, $a_1=1$, $a_2=3$, $a_3=3^2$, etc.

In other words, the k pieces are 1, 3, 9, 27...etc.

When $2n+1$ is not an exact power of 3, the problem of finding the values of $a_1 \ldots a_k$ is indeterminate, but the least value of k is such that $3^k > 2n+1$.

For example, take the case $n=44$ The value of k here is $k=5$.

In this case the five weights may either be

$$1, 3, 9, 27, 4$$
$$1, 3, 9, 26, 5.$$

The reason is that it is possible to choose $p_1 \ldots p_k$ so that $(p_k \geq 1)$

$1+\sum_{1}^{44} p_k\ (x^k+x^{-k})$ may be the product of five factors of the form $(1+x^a+x^{-a})$.

Question 702 (ii).

(MARTYN M. THOMAS) :—Prove that

$$\text{(ii)}\quad \int_0^\infty \sin x \log|\cos x|\frac{dx}{x}=\pi\log\frac{1}{\sqrt{2}}$$

Additional Solution by K. B. Madhava.

More generally, let $f(\xi)$ be an odd function of ξ and consider the integral

$$\int_0^\infty f(\sin x)\frac{dx}{x} \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

Splitting up the interval into parts,

$$\int_0^\infty = \int_0^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\pi} + \int_{\pi}^{\frac{3}{2}\pi} + \ldots \qquad \ldots \qquad \ldots \quad (2)$$

Now in the second interval put $y=\pi-x$.

third $\quad z=x-\pi \ldots \quad \ldots \quad \ldots \quad$ (A)

fourth $\quad u=2\pi-x$ and so on.

$$\text{Thus} \int_0^\infty f(\sin x)\frac{dx}{x}=\int_0^{\frac{\pi}{2}} f(\sin x)\frac{dx}{x}+\int_0^{\frac{\pi}{2}} f(\sin x)\frac{dx}{\pi-x}+\ldots$$

$$=\int_0^{\frac{\pi}{2}} f(\sin x)\left[\frac{1}{x}-\left(\frac{1}{x-\pi}+\frac{1}{x+\pi}\right)\right]+\ldots dx$$

$$=\int_0^{\frac{\pi}{2}} f(\sin x)\operatorname{cosec} x\, dx.$$

Thus if both integrals are convergent, and f is an odd function we shall have

$$\int_0^\infty f(\sin x)\frac{dx}{x}=\int_0^{\frac{\pi}{2}} f(\sin x)\frac{dx}{\sin x}$$

This converts the infinite range into a finite range, and all the expressions involved are circular functions, and often it will be found easier to evatuate the right hand side.

In this particular example, the transformation (A) leaves the function unaltered, except that the cos x changing sign, its logarithm introduces some infinities. To avoid this we put the modulus sign before cos x and read the question as written at the top of this paper.

The question therefore reduces to

$$\int_0^\infty \sin x \log|\cos x|\frac{dx}{x}$$

$$=\int_0^{\frac{\pi}{2}} \sin x \log|\cos x| \operatorname{cosec} x\, dx.$$

$$=\int_0^{\frac{\pi}{2}} \log|\cos x|\, dx.$$

$$=\frac{\pi}{2}\log\tfrac{1}{2},$$

$$=\pi\log\frac{1}{\sqrt{2}}.$$ [*Vide*: Bromwich: App. III, Exs. 11, 12].

Question 724.

(S. RAMANUJAN) :—Show that

(i) $$\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}\ldots\text{to } n \text{ terms}$$

$$=\tan^{-1}\frac{1}{1+2\cdot1^3}+\tan^{-1}\frac{1}{1+2\cdot2^3}+\tan^{-1}\frac{1}{1+2\cdot3^3}+\ldots\text{to } n \text{ terms};$$

(ii) $$\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+3)\sqrt{3}}+\ldots\text{to } n \text{ terms}$$

$$=\tan^{-1}\frac{1}{(\sqrt{3})^3}+\tan^{-1}\frac{1}{(3\sqrt{3})^3}+\tan^{-1}\frac{1}{(5\sqrt{3})^3}+\ldots\text{to } n \text{ terms}.$$

Solution by K. B. Madhava.

The question is wrongly printed; the first part ought to read as follows.

Show that

(i) $$\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}+\ldots\text{to } n \text{ terms}$$

$$=\tan^{-1}\frac{1}{1+2\cdot1^2}+\tan^{-1}\frac{1}{3(1+2\cdot3^2)}+\tan^{-1}\frac{1}{5(1+2\cdot5^2)}+\ldots\text{to } n \text{ terms};$$

The results can be established easily by Mathematical Induction; for, assuming it to be true for n, we shall prove it for $n+1$; when we have

$$S_{n+1} \equiv \tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}+\ldots$$

$$\tan^{-1}\frac{1}{4n+1}+\tan^{-1}\frac{1}{4n+3}\ [n+1 \text{ terms in number}]$$

$$=S_n+\tan^{-1}\frac{1}{4n+1}+\tan^{-1}\frac{1}{4n+3}-\tan^{-1}\frac{1}{2n+1}.$$

But

$$\tan^{-1}\frac{1}{4n+1}+\tan^{-1}\frac{1}{4n+3}=\tan^{-1}\frac{4n+2}{8n^2+8n+1},$$

and
$$\tan^{-1}\frac{4n+2}{8n^2+8n+1}-\tan^{-1}\frac{1}{2n+1}=\tan^{-1}\frac{1}{(2n+1)(8n^2+8n+3)}$$

$$=\tan^{-1}\frac{1}{(2n+1)[1+2\cdot(2n+1)^2]},$$

which is exactly the term to be newly added on the right side.

And to complete the proof, we can prove the result true for the first few cases.

When $n=1$; $\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{3}$.

When $n=2$; $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{6}{17}$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{57}=\tan^{-1}\frac{6}{17},$$

(ii) Similarly with the second result.

With analogous notation and method, S_{n+1} differs from S_n by

$$\tan^{-1}\frac{1}{(4n+1)\sqrt{3}}+\tan^{-1}\frac{1}{(4n+3)\sqrt{3}}-\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}$$

$$=\tan^{-1}\frac{(2n+1)\sqrt{3}}{12n^2+12n+2}-\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}$$

$$=\tan^{-1}\frac{1}{\sqrt{3}}\cdot\frac{1}{(2n+1)(12n^2+12n+3)}$$

$$=\tan^{-1}\frac{1}{[(2n+1)\sqrt{3}]^3},$$

which is precisely the term to be added when n is increased by unity.

The first few cases when $n=1$, 2 etc., being easily verified as before the proof by Induction is complete.

Question 729.

(K. PADMANABHULU, B.A.) :—If the earth were to break up into an indefinite number of fragments at any point in its course round the sun by any sudden explosion, prove that all the fragments meet again at the same point; and that at the middle of the interval between the explosion and junction all the pieces will be moving with equal velocities in parallel directions.

Solution by Martyn M. Thomas.

When the explosion takes place, the fragments are all scattered in different directions with an equal velocity, say v.

They describe elliptic paths with the sun at one focus.

The relation $v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$ shows that v, μ, r being the same for all these paths, a also must be the same.

Thus the periodic time which is $\frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}$ is also the same for all the paths.

Hence the fragments will again pass through the point at which the explosion took place, after the lapse of time $\frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}$

At the middle of the interval between explosion and junction, the several fragments will be situated at the diametrically opposite points of their respective orbits, and must be moving with the same velocity v, but in directions opposite and parallel to their initial directions Hence the theorem.

Question 736.

(R. SRINIVASAN, M.A.) :—Show that the common tangent to the nine-point and inscribed circle of the triangle ABC cuts the sides a, b, c in the ratios

$$\frac{a-b}{a-c},\ \frac{b-c}{b-a},\ \frac{c-a}{c-b}.$$

Solution by K. V. A. Sastri, B.A., M. M. Thomas, K. B. Madhava, V. Anantaraman and K. Appukuttan Erady, M.A.

The equation of the radical axis of two circles is

$$(t_1^2-t'^2_1)x+(t_2^2-t'^2_2)y+(t_3^2-t'^2_3)z=0,$$

where the t's are the lengths of the tangents from the vertices to the two circles. (See Milne : *Homogeneous Co-ordinates*, p. 111.)

But, for the nine-point circle and the incircle, the radical axis is, by Feuerbach's theorem, the common tangent. Hence its equation is

$$\left[\frac{bc\cos A}{2}-(s-a)^2\right]x+\left[\frac{ca\cos B}{2}-(s-b)^2\right]y + \left[\frac{ab\cos C}{2}-(s-c)^2\right]z=0,$$

which reduces to

$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0\,;$$

hence it cuts the sides of the triangle in the ratios

$$\frac{a-b}{a-c},\ \frac{b-c}{b-a},\ \frac{c-a}{c-b},$$

as given in the problem.

Additional Solutions by R. D. Karve, S. V. Venkatarayasastri, M. K. Kewalramani, M.A., and Chas. Saldanha.

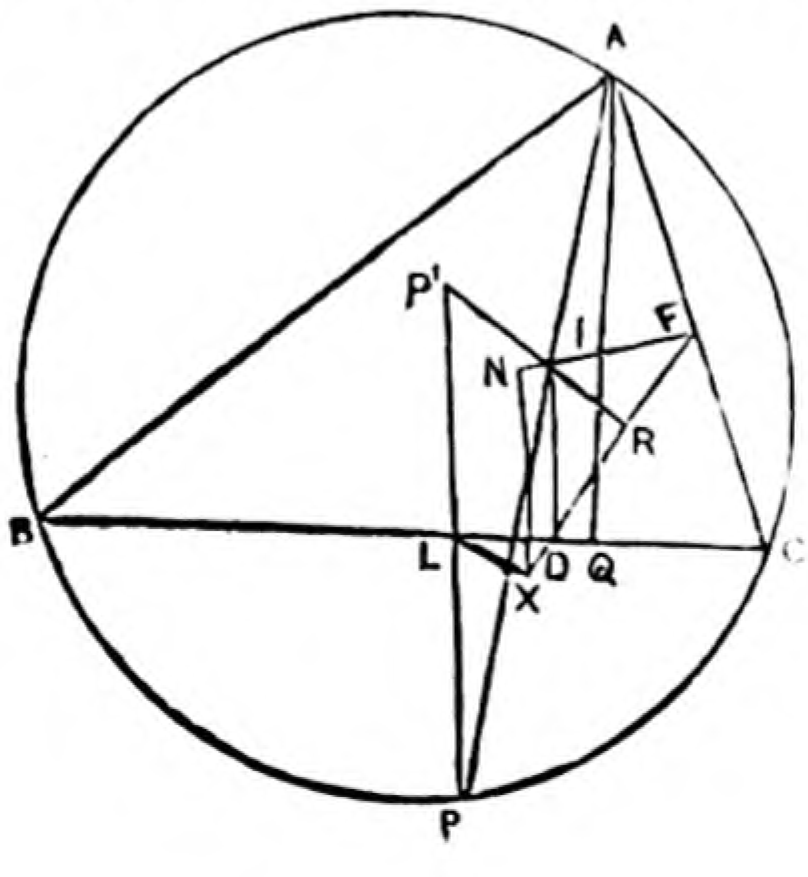
Question 743.

(N. Salva) :—Prove the following construction for the Feuerbach-point :

If AI meet the circumcircle in P and P′ be the reflection of P in BC, then the Feuerbach-point F is the reflection of D in IP′, where D is the point of contact of the incircle and BC.

Solution by K. J. Sanjana.

Draw NX the radius of the nine-point circle perpendicular to BC ; then X corresponds to D in the incircle, and XD (like NI) will go through F, the external centre of similitude. Draw IR perpendicular to DF to meet in P′ the line PL perpendicular to BC.



Then $\angle PP'I = \angle DIR = \angle RDQ = \angle LDX$; also $\angle P'PI = \angle IAQ = \frac{1}{2}(C-B)$, and $\angle DLX = \angle DFQ = \frac{1}{2}\angle LFQ = \frac{1}{2}(C-B)$ from the nine-point circle. Thus the Δs P′IP and XLD are similar, and $PP' : PI = LD : LX$.

Now $PI = 2R\sin\frac{A}{2}$; $LD = \frac{1}{2}(c-b) = 2R\sin\frac{A}{2}\sin\frac{C-B}{2}$; and $LX = R\sin\frac{C-B}{2}$, from the nine-point circle. Thus, we get

$$PP' = 4R\sin^2\frac{A}{2} = 2PL,$$

so that P′ is the reflection of P in BC. Hence the theorem.

It will similarly be found that the point of contact of the nine-point with the first scribed circle is the reflection in $P'I_1$ of D_1, the point of contact of the ex-circle with BC.

Question 744.

(N. Salva) :—If ABCD, AB′ CD′ be two harmonic ranges, such that D′ is the midpoint of CD, prove that $BC^2 = BB'.BD$.

Solution by Lakshmishankar Bhatt and K. J. Sanjana.

Referring to A as origin, let B, C, D be denoted by x_1, x_2, x_3 respectively, so that $2x_1x_3 = x_2(x_1+x_3)$...(i)

Now D′ will be $\frac{1}{2}(x_2+x_3)$, so that if $AB' = y$

$$2y.\tfrac{1}{2}(x_2+x_3) = x_2\{y + \tfrac{1}{2}(x_2+x_3)\},$$

or

$$y = (x_2^2 + x_2x_3)/2x_3.$$

Hence

$$y(x_3-x_1) = \frac{1}{2x_3}\left\{x_2^2x_3 - x_2^2x_1 + x_2x_3^2 - x_1x_2x_3\right\}$$

$$= \frac{1}{2x_3}\left\{x_2x_3(x_2+x_3-x_1) - x_2(2x_1x_3 - x_2x_3)\right\}, \text{ from (i)}$$

$$= \tfrac{1}{2}\{x_2^2 + x_2x_3 - x_2x_1 - 2x_2x_1 1 + x_2^2\}$$

$$= \tfrac{1}{2}\{2x_2^2 - 3x_2x_1 + 2x_1x_3 - x_2x_1\}, \text{ from (i)}$$

$$= x_2^2 - 2x_2x_1 + x_1x_3$$

$$= (x_2-x_1)^2 + x_1(x_3-x_1);$$

thus finally $(x_2-x_1)^2 = (x_3-x_1)(y-x_1)$

or

$$BC^2 = BD.BB'.$$

There is a slight mistake in the example as originally given.

Additonal Solution by S. V. Venkatachala Iyer.

Question 745.

(V. ANANTARAMAN) :—How can 49 diamonds whose values are in A. P., be divided among 7 persons so that each may get the same number of diamonds, the total value of the diamonds got by each being the same.

Solution by R. D. Karve, S. Venkataraya Sastri, S. V. Vekatachala Iyer and others.

Let the values of the diamonds be denoted respectively by

$$a+d,\ a+2d, \ldots\ldots\ldots\ldots a+49d.$$

Then the total value of the diamonds that each of the 7 persons should get is denoted by $7a+175d$.

The question thus reduces to selecting 7 groups of 7 numbers each, from the numbers 1 to 49 so that the sum of each group may be 175. This may be done by taking the groups of numbers, either along the rows or along the columns in a magic square filled with numbers 1 to 49 as represented below.

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

Question 758.

(S. MALHARI RAO, B. A.) :—If the integers x, y, z represent the sides of a right-angled triangle ; and x, z, are primes greater than 5, show that y is a multiple of 60 ; and that $x+z=1800$, when $y=1740$.

Solution (1) *by K. Appukuttan Erady, M.A. and* '*Q*' ;
(2) *by K. J. Sanjana, M.A. and R. D. Karve, M.A.*

(1) We have $$x^2+y^2=z^2.$$

$\therefore$ $$x^2=z^2-y^2=(z+y)(z-y).$$

Hence since x is prime

$$z-y=1 \text{ and } z+y=x^2.$$

Thus when $y=1740$, $z=1741$, $x^2=3481$, $x=59$ and $x+z=1800$.

Again since $z-y=1$ and $z+y=x^2$, $2y=x^2-1$.

Now x being prime, $x-1$ & $x+1$ are two consecutive even numbers

$\therefore$ x^2-1 contains 8 as a factor.

Since $(x-1)\,x\,(x+1)$ contains 3 as a factor, and x is prime, x^2-1 contains 3 as a factor.

Again $(x-2)(x-1)x(x+1)(x+2)$ is divisible by 5; but $x^2-4=2y-3=2z-5$, and z is a prime greater than 5;

$\therefore$ x^2-1 contains 5 as a factor.

Hence x^2-1 contains $8\times3\times5$ as a factor. That is y is multiple of 60.

(2) Evidently x, y, z must be of the form

$$m^2-n^2,\ 2mn,\ m^2+n^2$$

Since x and z are primes, y is of the form $2mn$ and x, z of the forms m^2-n^2 and m^2+n^2, and

$$(2mn)^2=(m^2+n^2)^2-(m^2-n^2)^2.$$

Now $(m^2+n^2)^2-1$ and $(m^2-n^2)^2-1$ are each divisible by 3 by Fermat's Theorem. Hence their difference (the right hand side above) is divisible by 3,

$\therefore$ $2mn$ is divisible by 3.

Again all primes (>2) being of the form $4k\pm1$, their squares have the form $8p+1$ and the difference of squares of primes is therefore divisible by 8,

$\therefore$ $(2mn)^2$ is divisible by 8 and hence by 16 being a square,

$\therefore$ $2mn$ is divisible by 4.

Also all squares are of the forms $5k$, or $5k\pm1$. Hence either m^2 or n^2 must be of the form $5k$, otherwise either m^2+n^2; or m^2-n^2 would be composite. Hence $2mn$ is divisible by 5.

$\therefore$ Thus $2mn$ is a multiple of $3\times4\times5$ or 60.

Again, if $y=1740$, $2mn=1740$, and $mn=870$. Obviously, since m^2-n^2 *i.e.* $(m+n)(m-n)$ is to be a prime, $m-n$ must be 1.

Hence $m=30$ and $n=29$, and $x+z=2m^2=1800$.

QUESTIONS FOR SOLUTION.

789. (K. J. Sanjana, M. A.) :—P, Q, R, S are four conormal points on an ellipse whose centre is C and axes $2a$ and $2b$, O being the point of concurrence of the normals. If x_r, y_r $(r=1, 2, 3, 4)$ denote the centres of the circles Q R S, P R S, P Q S, P Q R respectively, prove that $\Sigma x=$ the abscissa of O, $\Sigma y=$ the ordinate of O, and that each centre lies on an ellipse whose centre is at the mid point of CO and whose axes are

$$\frac{a^2-b^2}{2a}, \frac{a^2-b^2}{2b}.$$

[Suggested by Mr. Neville's Question No. 788.]

790. (K. J. Sanjana, M. A.) :—Integrate the equation

$$(a^2+x^2)\frac{d^2y}{dx^2}+2(m+1)x\frac{dy}{dx}+m(m+1)y=f(x).$$

[Two particular integrals when $f(x)$ is absent are

$$\int_0^\infty e^{-az}\cos zx.z^{m-1}\,dz, \int_0^\infty e^{-az}\sin.xz.z^{m-1}\,dz.]$$

791. (K. Appukuttan Erady, M.A.) : —The centres of three circles of radii a, b, c form a triangle of sides l, m, n and area Δ. If $(r_1\, r_1')$, (r_2, r'_2) (r_3, r_3') and (r_4, r_4'), be the radii of the four pairs of circles tangential to the three circles (the circles belonging to any pair being inverses of each other with respect to the common orthogonal circle of the three original circles), show that

$$\frac{1}{r_1 r_1'}+\frac{1}{r_2 r_2'}+\frac{1}{r_3 r_3'}+\frac{1}{r_4 r_4'}$$
$$=\frac{16(\Delta^2-\Sigma a^2l^{2})}{\Sigma l^2(a^2-b^2)(a^2-c^2)-\Sigma m^2n^2(b^2+c^2)+\Sigma a^2l^4+l^2m^2n^2}.$$

792. (K. Appukuttan Erady, M.A.) :—With the usual notation in elliptic functions , show that if $\alpha+\beta+\gamma+\delta=0$,

(1) k'^2 (sn α sn $\beta-$sn γ sn δ)

$=$cn α cn β dn γ dn$\delta-$cn γ cn δ dn α dn β ;

(2) cn α cn $\beta-$cn γ cn $\delta=$

sn α sn β dn γ dn $\delta-$sn γ sn δ dn α dn β ;

(3) $\frac{1}{k^2}$ (dn α dn $\beta-$dn γ dn δ)

$=$sn α sn β cn γ cn $\delta-$sn γ sn δ cn α cn β.

793. (Martyn M. Thomas, M. A.) :—Two stars rise together, and are observed to come simultaneously over a vertical λ degrees west of the meridian. Show that they must have risen $\frac{1}{15}$ cos^{-1} ($-$sin$^2\lambda$) hours before; and that the latitude of the place is tan^{-1} (sinλ).

794. (MARTYN M. THOMAS, M.A.):—The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ moves in such a manner that the centroid of the triangle ABC always lies on the surface $x^{-2}+y^{-2}+z^{-2}=9$, A, B, C being the points where the plane meets the rectangular axes of co-ordinates. Prove that the Nine-points centre of ABC will lie on the surface

$$x\sqrt{4x^2-1}+y\sqrt{4y^2-1}+z\sqrt{4z^2-1}+2(x^2+y^2+z^2)=1.$$

795. (V. M. GAITONDE):—Prove that

$$\tan\frac{3\pi}{11}+4\sin\frac{2\pi}{11}=\sqrt{11}$$

796. (M. K. KEWALRAMANI, M.A.):—Prove that

$$\underset{x=0}{\mathrm{Lt}}\left[\frac{d^{2n}}{dx^{2n}}\left(x\cot x\right)^{2n+1}\right]=(-1)^n(2n)\,!$$

797. (K. S. KARPUR):—If two sides of a given polygon touch each a fixed circle, prove that the remaining sides also touch each a fixed circle.

798. (S. SENGODAIYAN):—Establish the identity

$$S(n,r)=n_r\cdot c_r-n_{r-1}\cdot c_{r-1}S(n-r,1)+n_{r-2}c_{r-2}S(n-r+1,2)$$
$$-+\ldots(-1)^rS(n-1,r),$$

where n_p denotes $n(n-1)\ldots(n-p+1)$, c_p denotes the number of combinations of n things p together, and $S(x,y)$ denotes the sum of the products of the first x natural numbers y at a time.

799. (S. MALHARI RAO.):—Find three primes in A. P. such that the sum of their squares is 35427. 37, 97, 157.

800. (S. MALHARI RAO.):—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $(.\dot{a}\,b\,c\,\dot{d})$ is 50, provided $a+c=b+d=9$.

801. (S. KRISHNASWAMI AIYANGAR):—

$$\text{If } a_n=1-\frac{3^n}{3\,!}+\frac{5^n}{5\,!}-\frac{7^n}{7}+\ldots\ldots$$

and

$$-b_n=\frac{2^n}{2\,!}-\frac{4^n}{4\,!}+\frac{6^n}{6\,!}-\ldots\ldots$$

prove that

$$\text{(i) } a_n+a_{n+1}\log p+a_{n+2}\frac{(\log p)^2}{2\,!}+a_{n+3}\frac{(\log p)^3}{3\,!}+\ldots\ldots$$
$$=p-3^n.\frac{p^3}{3\,!}+5^n.\frac{p^5}{5\,!}-\ldots\ldots$$

(ii) $\frac{1}{2}\sin(e^x+e^{-x})$

$$=\sum_0^\infty\left\{a_{2n}b_0+{}^{2n}C_2a_{2n-2}b_2+\ldots\ldots,+a_0b_{2n}\right\}\frac{x^{2n}}{2n!}$$

802. (S. Krishnaswami Aiyangar) :—Prove that

(i) $\displaystyle\sum_1^\infty\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2\frac{1}{n}=4\left(\frac{4}{\sqrt{\pi}}-\sqrt{\pi}\right)$

(ii) $\displaystyle\sum_{n=0}^\infty\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\cdot\frac{1}{(2m+2n+1)(m+n+1)}$

$$=\pi\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}-2.\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}\right\}$$

803. (Selected) :—Prove that the Rectangular Hyperbola $x^2-y^2-4ax\cos^3\alpha+4ay\sin^3\alpha+3a^2\cos 2\alpha=0$ osculates its envelope.

804. (Selected) :—In the curve whose intrinsic equation is $\dfrac{ds}{d\psi}=a$ sec. 3ψ, show that the rectangle under the distances of any point from the foci of the osculating conic is constant.

805 (E. H. Neville).—With the notation of Questions **412** and **761**, but allowing the angles α, β to be variable, shew that the necessary and sufficient condition for the current point of the α—evolute of the β—evolute to be the current point of the β—evolute of the α—evolute is that either α or β is a right angle or that the difference between α and β is constant.

806. (S. Narayana Aiyar, M. A.) :—Demonstrate the following :—

(1) $$\frac{d}{dx}\frac{\Gamma(x+a)}{\Gamma(x+b)}=\frac{\Gamma(x+a)}{\Gamma(x+b)}\cdot\left\{\frac{a-b}{x+b}-\frac{1}{2}\frac{(a-b)(a-b-1)}{(x+b)(x+b+1)}+\frac{1}{3}\frac{(a-b)(a-b-1)(a-b-2)}{(x+b)(x+b+1)(x+b+2)}-\ldots\right\}$$

(2) $$\frac{d}{dx}\frac{\Gamma(a-x)}{\Gamma(b-x)}=\frac{\Gamma(a-x)}{\Gamma(b-x)}\cdot\left\{\frac{b-a}{a-x-1}-\frac{1}{2}\frac{(b-a)(b-a+1)}{(a-x-1)(a-x-2)}+\frac{1}{3}\frac{(b-a)(b-a+1)(b-a+2)}{(a-x-1)(a-x-2)(a-x-3)}-\ldots\right\}$$

807. (F. H. V. Gulasekaran) :—Construct a trapezium having given the lengths of diagonals and the oblique sides.

List of Periodicals Received.

(From 16th July to 15th September 1916.)

1. Annals of Mathematics, Vol. 17, No. 4, June 1916.
2. Astrophysical Journal, Vol. 43, Nos. 4 & 5, May and June 1916.
3. Bulletin of the American Mathematical Society, Vol. 22, Nos. 9 & 10 June, July & August 1916.
4. Bulletin des Sciences Mathematics, Vol. 40, May, June & July 1916.
5. L'Intermediaire des Mathematiciens, Vol. 23, Nos. 5 & 6, May & June 1916.
6. Liouville's Journal, Vol. 1, No. 4.
7. Mathematical Gazette, Vol. 8. Nos. 123 and 124, May and July 1916 (3 Copies).
8. Mathematical Questions and Solutions, Vol. 2, Nos. 1 & 2, July & August 1916. (5 Copies).
9. Mathematics Teacher, Vol. 8, No. 4, June 1916.
10. Messenger of Mathematics, Vol. 45 and 46 Nos. 12 and 1, April and May 1916.
11. Philosophical Magazine, Vol. 32, Nos. 187 and 188, July and August 1916.
12. Popular Astronomy, Vol. 24, Nos. 6 & 7, June, July, August and Septem- 1916. (3 Copies)
13. Proceedings of the London Mathematical Society, Vol. 15, No. 4, August 1916.
14. Proceedings of the Royal Society of London, Vol. 92, No. 642 July 1916.
15. Quarterly Journal of Mathematics, Vol. 47, No 2, July 1916.
16. Transactions of the American Mathematical Society, Vol. 17, No. 3 July 1916.
17. Transactions of the Royal Society of London, Vol. 216, Nos. 545, 546, 547 and 548 and Vol 217, Nos. 549 and 550.
18. The Tohoku Mathematical Journal, Vol. 9, No. 4, June 1916, Vol. 10, Nos. 1 and 2, August 1916.
19. American Mathematical Monthly, Vol. 23, Nos. 5 and 6, May and June 1916.

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CONTENTS.

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PROGRESS REPORT.

The following gentlemen have been elected members :—

1. *Mr. Hiralal R. Kapadia, B.A.*—Asst. Teacher, Bhatia High School, 43, Navi Chawl, Bhagatwadi, Bhuleshwar, Bombay (at concessional rate);

2. *Mr. A. Gurumurti, B.A., L. T.*—Head Master, Board High School, Kavali (Nellore District.)

2. The attention of the members of the Society is drawn to the fact that all arrangements are being completed for holding the proposed meeting of the Society at Madras on 26th, 27th and 28th of December 1916. The detailed programme for this will be posted to our members shortly and the Committee most earnestly expects that the members will try their utmost to make this meeting a success.

3. *The Calcutta University Calendar, Part I* for 1916 has been received for the Library.

POONA,
30th Nov. 1916.

D. D. KAPADIA,
Hon. Joint Secretary.

The Homographic Function as an Operator.

By R. Vythynathaswamy.

[References: Forsyth:—*Theory of Functions of a Complex Variable*, Chs. XIX, XXI].

§ I.

The most general *algebraical* one-to-one correspondence between points on a line is given by the transformation

$$x' = S(x) = \frac{ax+b}{cx+d}.$$

The function S may be regarded as an operation which carries the point x to the point x'. If S_1, S_2 are to operations of this form, $S_2(S_1)$ is an operation of the same form; so that, the totality of these operations forms a 'group.' very operation S has a unique *inverse* operation of the same form, so that if $x' = S(x)$, we may write $x = S^{-1}(x')$. The *identical* operation of the group (corresponding to the transformation $S(x) = x$, may be denoted by the symbol 1.

If $S^2 = 1$, then S is an involution.

If S is not the identical transformation, the equation $S(x) = x$, determines two points which are *the fixed points* of S. Evidently these will also be the fixed points of any power of S or its inverse.

Theorem (1): *If a correspondence carries two distinct points into each other, it is an involution.*

Let $x = S(y)$, and $y = S(x)$.

Then $x = S(y) = S^2(x)$.

Since x, y are distinct, x is not a fixed point of S and therefore not a fixed point of S^2.

$\therefore$ $S^2 = 1$ identically;

i.e. S is an involution.

Theorem (2): *If I and SI be both involutions then IS is an involution and I contains the fixed points of S.*

For we have $(SI)^2 = 1$.

i.e. $SISI = 1$.

$\therefore$ $ISISI = I$.

$\therefore$ $(IS)^2 I = I$.

$\therefore$ $(IS)^2 = 1$.

Hence IS is an involution, (More generally if S_1, S_2 is an involution $S_2 S_1$ is also an involution. See § II, below.)

Also if t_1, t_2 be the fixed points of S

$$(SI)^2 t_1 = t_1.$$

$$\therefore \quad SI(t_1) = I^{-1} S^{-1}(t_1) = IS^{-1}(t_1) = I(t_1)$$

Hence $I(t_1)$ is a fixed point of S.

Now if $I(t_1)$ were equal to t_1 so that t_1, t_2 are the fixed points of I, then the fixed points of both SI and IS would be t_1, t_2. Hence, since these are involutions $SI \equiv IS$.

$\therefore \qquad S^2 = SI.IS = SI.SI = (SI)^2 = 1.$

$\therefore \qquad$ S is an involution

Excluding this case, we have $I(t_1) = t_2$ or I contains the pair of fixed points of S.

Cor. *Conversely if I be an involution containing the fixed points of* S, *SI is an involution.*

For $$SI(t_1) = S(t_2) = t_3$$

and $$SI(t_2) = S(t_1) = t_1.$$

Hence by Theorem (1) SI is an involution.

Combining the theorem and the Corollary we see that the correspondence S^2 can be exhibited in an infinite number of waysas the product of two involutions SI and IS. We shall presently shew that any correspondence can be exhibited as a square so that this result is general.

Theorem (3): *If two correspondences* S_1 S_2 *are permutable, either they have the same fixed points, or each of them is an involution containing the fixed points of the other.*

Let t_1, t_2 be the fixed points of S_2. Since S_1, S_2 are permutable by hypothesis, we have

$$S_2S_2(t_1) = S_2S_1(t_1)$$

$$\therefore \quad S_1(t_1) = S_2S_1(t_1),$$

so that $S_1(t_1)$ is a fixed points of S_2.

Hence, either $S_1(t_1) = t_1$ and $S_1?t_2) = t_2$

or $$S_1(t_1) = t_2 \text{ and } S_1(t_2) = t_1.$$

In the former case, S_1 and S_2 have the same fixed points. In the latter case, S_1 is an involution containing the fixed points of S_2. Since we might have assumed t_1, t_2 to be the fixed points of S_1 instead of S_2, it follows simultaneously that S_2 is an involution containing the fixed points of S_1. We further note that in this case $S_1 S_2$ $(= S_2S_1)$ is an involution and contains the fixed points of both S_1 and S_2, so that S_1, S_2 S_1S_2 *are three involutions each of which is determined by the fixed points of the other two.*

For $$S_1S_2(t_1) = S_1(t_1) = t_2,$$

and $$S_2S_1(t_2) = S_2(t_1) = t_1.$$

Hence by Theorem (1) $S_1 S_2$ is an involution evidently containing the fixed points of both S_1 and S_2.

Cor. (1): The converse of the theorem is true. That of the first part is proved under Theorem (5) below. We prove here the converse of the second part, *viz. Two involutions each containing the fixed points of the other are permutable and their product is the involution containing their fixed points.*

For if I_1, I_2 be the two involutions, the equation $I_1I_2 = I_2I_1$ is true for each of the fixed points of I_1 and I_2. Now, if $S(x) = S(x)$ for *three* distinct values of x, then $S = S_1$ identically.

Hence $$I_1I_2 \equiv I_2I_1$$

From this it is easy to see that three involutions whose product is unity would be represented in the plane by the sides of a triangle self-conjugate *w. r. t* the fundamental conic. (See Note on Involution and (1, 1) correspondence J.I.M.S., February 1916).

As an example consider the involutions

$$I_1 = (a_1\, a_2,\, a_3\, a_4),\ I_2 = (a_1\, a_3,\, a_2\, a_4),\ I_3 = (a_1\, a_4,\, a_3\, a_2)$$

It is at once verified that $I_1\, I_2\, I_3 = 1$.

Hence *the harmonic triangle of an inscribed quadrangle is self-conjugate.*

Cor. (2): *If the product of three involutions I_1, I_2, I_3 is an involution I, then all the four involutions have a common pair.*

For $I_1 . I_2\, I_3 = I$ = an involution.

$\therefore$ I_1 contains the fixed points of I_2I_3 (Theorem (2))

Now the pair of fixed points of the product of two involutions is simply their common pair. Hence I_1, I_2, I_3 have a common pair. Evidently, I also contains this pair.

Theorem (4). *If* S *be a correspondence and p, q any two points, the involution $I \equiv [p, S(q)\,;\ q, S(p)]$ always contains the fixed points of* S.

For $IS(p) = q$ and $IS(q) = p$.

Hence by Theorem (1) IS is an involution

$\therefore$ I contains the fixed points of S (Th. 2).

Cor. Let $S \equiv \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$ *i.e.* Let S carry p, q, r respectively into $p'\ q'\ r'$. From the present theorem we see that the involution $I_1 = (pq', p'q)$, $I_2 = (qr', q'r)$ $F_3 = (rp', r'p)$ have a common pair, *viz.* the pair of fixed points of S.

This is evident otherwise. For, $I_1I_4I_3(p) = p'$ and $I_1\ I_2\ I_3\ (p') = p$. Hence $I_1I_2I_3$ is an involution.

$\therefore$ $I_1I_2I_3$ have a common pair [Th. (3). Cor. 2]. Transforming this result to the plane we have Brianchon's theorem.

Theorem (5). *If two correspondences have the same fixed points they are permutable.*

Let S_1, S_2 be the two correspondences.

Let $x_1 = S_1(x)$, $x_2 = S_2(x_1)$, $x_3 = S_2(x)$, $x' = S_1(x_3)$;

so that S_1 is $\begin{pmatrix} x \; x_3 \\ x_1 \; x' \end{pmatrix}$ and S_2 is $\begin{pmatrix} x_1 \; x \\ x_2 \; x_3 \end{pmatrix}$

By Theorem (4) the involutions $(x_1x_3,\ xx')$, $(x_1x_3,\ xx_2)$ contain the fixed points of S_1 and S_2 respectively. If these fixed points are the same, the involutions become identical since they have another common pair x_1x_3.

Hence $$x' = x_2. \text{ i.e. } S_1S_2(x) = S_2S_1(x).$$

Thus S_1, S_2 are permutable.

§ 11.

The parameter of a correspondence. The fixed points may be supposed to define the *position* of the correspondence in the line. We now proceed to find a function which is an appropriate quantitative measure of a correspondence S; *i.e.*, which satisfies the relation

$$f(S^n) = \{ f(S) \}^n.$$

By a well-known theorem in the Theory of Linear Transformation the determinant of the matrix of S^n is the n^{th} power of the determinant of the matrix of S. But the difficulty in taking Δ the determinant of S $\left(=\dfrac{ax+b}{cx+d}\right)$ for our function, is that Δ depends on the absolute values of $a\, b\, c\, d$, while S depends only on their ratios.

Let S be reduced to the form $\dfrac{kx+t^2}{x+k}$, so that the fixed points are $\pm t$. Let S′ be another correspondence having the same fixed points, so that

$$S'(x) = \frac{k'x+t^2}{x+k'}.$$

Let $$S'' = S'S = \frac{k''x+t^2}{x+k''}.$$

$$\text{Then } k'' = S''(\infty) = S'S(\infty) = S'(k) = \frac{kk'+t^2}{k+k'}.$$

$$\therefore \qquad \frac{k''}{t} = \frac{kk'+t^2}{t(k+k')}; \text{ so that } \frac{k''+t}{k''-t} = \frac{k'+t}{k'-t}\cdot\frac{k+t}{k-t}.$$

Hence if $\dfrac{k+t}{k-t}$ be termed the parameter of S (called 'multiplier' by Forsyth) we have the theorem:—

The parameter of the product of two correspondences having the same fixed points is the product of their parameters.

Hence: *if* λ *be the parameter of* S, λ^n *is the parameter of* S^n.

The parameter λ is verified to be the cross-ratio of the fixed points of S and any pair of correspondents of S.

It can also be shewn that the parameter of $S\left(=\frac{ax+b}{cx+d}\right)$ is the ratio of the latent roots of the matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, *i.e.*, of the roots of the equation

$$\begin{vmatrix} a-\rho & b \\ c & d-\rho \end{vmatrix}=0.$$

The value of the parameter in terms of $a\,b\,c\,d$ is given by

$$\lambda=\frac{(a+d)^2-2\Delta+(a+a)\sqrt{(a+d)^2-4\Delta}}{2\Delta}$$

where $\qquad \Delta=ad-bc.$

It must be noted that the definition of the parameter involves a definite order of the fixed points. If this order is changed the parameter will be changed into its reciprocal.

(1) A correspondence is uniquely determined if the fixed points and the parameter be known.

(2) Since λ^n is the parameter of S^n when n is integral, *we may define* S^n, *when* n *is not integral, to be the determinate correspondence which has the same fixed points as* S *and whose parameter is* λ^n.

The present theorem then proves that $S^m . S^n = S^{m+n}$ for all values of m and n.

If S is given in the form $\frac{kx+t^2}{x+k}$, the present theorem gives at once

$$S^n(x)=\frac{(\lambda^n+1).\,xt+t^2(\lambda^n-1)}{(\lambda^n-1)x+(\lambda^n+1)\,t}, \qquad \left(\lambda=\frac{k+t}{k-t}\right)$$

for all values of n.

(3) *Periodic correspondences.*

A correspondence is said to have a period n, when its n^{th} power is the identical correspondence. The parameter of such a correspondence would be an n^{th} root of unity; (k ($=S(\infty)$) being infinite for the identical correspondence, its parameter is 1). The number of correspondences of period n and no smaller period, with given fixed points, is equal to the number of integers less than n and prime to it. The forms of these are easily found as well as the condition that a correspondence may have the period n.

(4) *Uniparametric correspondences.*

This is the class of correspondences which are not periodic but are *nearly periodic*. The modulus of the parameter of such a correspondence S would be unity and the amplitude would be incommensurable with π. What is meant by saying that S is nearly periodic, is that an integer n can always be chosen so that S^n is as nearly equal to the identical correspondence as we please. As a matter of fact the set of points in a line defining segments commensurable with a given segment has every point as a limit-point.

We note that every integral power of a uniparametric correspondence is uniparametric and that a correspondence the modulus of whose parameter is unity, is uniparametric unless it is periodic.

(5) *The correspondence $S_1^{-1} SS_1$ is termed the transform of S by S_1; and the parameter of any transform of S is equal to the parameter of S.*

Let $T = S_1^{-1} SS_1$ and let x, y be the fixed points of S. The fixed points of T are $S_1^{-1}(x)$, $S_1^{-1}(y)$.

We have,

$$\begin{aligned}
\text{parameter of } T &= \text{cross-ratio } \{ S_1^{-1}(x), S_1^{-1}(y), p, S_1^{-1}SS_1(p) \} \\
&= \text{cross-ratio } \{ x, y, S_1(p), SS_1(p) \} \\
&= \text{cross-ratio } \{ x, y, q, S(q) \} \\
&= \text{parameter of S.}
\end{aligned}$$

As a corollary it follows that the parameter of S_1S_2 (S_1 and S_2 not having the seme fixed points) is equal to the parameter of S_2S_1, though neither of these would be equal to the product of the parameters of S_1 and S_2. [Theorem (2) is an example of this property].

For $S_2S_1 = S_1^{-1}(S_1S_2)S_1 =$ a transform of S_1S_2.

Hence follows the interesting property: *that the parameter of the product of a number of correspondences is the same so long as the cyclical order of the product is the same.*

§III.

Sub-groups of the general Homographic Group :

(1) *A continuous sub-group of the general homographic group must either consist of all correspondences having the same fixed points, or of all correspondences having a single point as one of their fixed points.*

Let S_1 belong to the continuous sub-group K. Then $S_1 \pm m$, where m is an integer belongs to K.. If K contains no other correspondence with the same fixed points as S_1, then K cannot be a *continuous* sub-group. For, we cannot find an *integer* n so that

$$S_1^n(x) = x + dx$$

i.e., $S_1^n = 1 + \delta$ (where S_1 is the parameter of S_1 and dx and δ are arbitrarily small).

Hence K does not contain the infinitesimal transformation.

It is easy to see that this necessarily leads to the conclusion that K must contain *all correspondences* having the same fixed points as S_1.

[It might appear that this reasoning would fail if S_1 were uniparametric. This exception would not however arise, if we insist that *a continuous group must contain not merely some but all possible infinitesimal transformations* i.e., must contain a transformation which carries a point x into any *assigned* near point $x+dx$. On this understanding it is easy to shew that the group $S_1 \pm m$ (S_1 being uniparametric) is discontinuous. For, though we can choose n so that S_1^n is an infinitesimal transformation, yet there are an infinite number of infinitesimal transformations with the same fixed points, which are not expressible in the form S_1^n; for example, the infinitesimal *periodic* transformations are not so expressible. As a matter of fact the set of points on the unit circle in the Argand diagram, representing the parameters of *all* uniparametric correspondences, is a point-wise discontinuous set; the points of discontinuity (the points whose amplitudes are rational multiples of π) forming evidently an enumerable aggregate.]

Let now, K contain, if possible, a correspondence S_2 not having the same fixed points as S_1. We have to consider two cases:—

Case I.—S_2 has both its fixed points different from those of S_1.

Repeating the argument above we shew that every correspondence P_2 having the same fixed points as S_2, belongs to K. Hence if P_1 has the same fixed points as S_1, every correspondence of the form $(P_1P_2)^t$ belongs to K.

It can be easily shewn that, by properly choosing the parameters P_1 and P_2, we can make $(P_1P_2)^t$ have any two given points as its fixed points; and by properly choosing t we can make it have any parameter we choose; thus K coincides with the general homographic group and is not a sub-group.

The supposition of *Case I* is therefore impossible.

Case II.—S_2 has one fixed point, L, common with S_1. We shew, on the same principle as before that K, if it is to be continuous, must contain all correspondences one of whose fixed points is L.

Hence the theorem is proved.

We note that in the first case (when Kcontains no correspondence (S_2), K is one—dimensional *i.e.* the form of its most general infinitesimal transformation contains one arbitrary constant. In the latter case K is two-dimensional.

(2) The group $S^{-m}.S^{-m+1}...1\ S\ S^2$ is a discontinuous subgroup of finite or infinite order according as S is or is not periodic.

There is a remarkable type of *a cyclic* subgroup of finite order 4. This is the group consisting of the identical correspondence and three involutions each of which contains the fixed points of the other two (*Cf.* Th. (3)). The existence of this type of groups is closely related to (and may in fact be used to prove) the possibility of algebraic solution of the general biquadratic. It is fairly certain that this is the only possible type of acyclic subgroups of finite order. I have not however seen it proved anywhere.

§ IV.

The Theory of Distance.

(1) The well-known projective definition of distance may be stated concisely in terms of a fundamental correspondence S.

Let the fixed points of S be $t_1\ t$; (S is to be neither periodic nor uniparametric.)

Let
$$\begin{pmatrix} t_1 & t_2 & p \\ t_1 & t_2 & q \end{pmatrix} = S^{\lambda}.$$

(Note that any two correspondences with the same fixed points can be expressed as powers of one another).

The distance pq may be defined to be the index λ. This definition is in conformity with our ordinary ideas.

For if
$$\begin{vmatrix} t_1 & t_2 & p \\ t_1 & t_2 & q \end{vmatrix} = S^{\lambda}, \text{ then } \begin{vmatrix} t_1 & t_2 & q \\ t_1 & t_2 & p \end{vmatrix} = S^{-\lambda}$$
so that $pq+qp=\lambda-\lambda=0$.

Also if
$$\begin{vmatrix} t_1 & t_2 & p \\ t_1 & t_2 & q \end{vmatrix} = S^{\lambda}, \begin{vmatrix} t_1 & t_2 & q \\ t_1 & t_2 & r \end{vmatrix} = S^{\mu},$$
then
$$\begin{vmatrix} t_1 & t_2 & p \\ t_1 & t_2 & r \end{vmatrix} = S^{\lambda}.S^{\mu} = S^{\lambda,+\mu}$$
so that $pq+qr=pr$.

Further the parameter of S^n is the n^{th} power of the parameter of S and therefore tends either to zero or infinity as $n \to \infty$.

Hence $S^{\infty}(x)=t_1$ or t_2, where x is any point.

Thus t_1, t_2 are at an infinite distance from all other points.

The distance as thus defined will be multiple-valued, for we can always find x from the equation
$$S^x=1, \ i.e., \ s^x=1$$

where s is the parameter of S. x as thus determined will not in general be real. But if S is negative

$$x=\frac{2ik\pi}{\log s}=\frac{2k}{\log(-s)}=\text{a real number}.$$

For example, if $s=-e^{a/\pi}$, the distance is multiple-valued to the extent of a multiple of $2\pi/a$. This corresponds to distance on the sphere.

(2) *This definition of distance is the most general possible one in conformity with our spatial ideas.* (Klein.)

Distance is perceived as something connected with two points, which is preserved by a class (C) of homographic transformations which are the congruent transformations. The class (C) is a continuous one-dimensional subgroup of the homographic group. For, (C) is a group since if C_1 and C_2 severally preserve distance, C_1C_2 preserves distance and (C) is continuous and one-dimensional since a unique C belonging to (C) can be found to carry any arbitrary point into any other. Hence by § III (1), (C) consists of all correspondences having the same fixed points.

If the C's be correspondences belonging to C and (pq) represent the distance pq, we have

$$\begin{aligned}(pC(p)) &= \{ C_1(p)C_1C(p) \}, \text{ since } C_1 \text{ preserves distance}\\ &= \{ C_1(p)CC_1(p) \}, \text{ since } C_1 \text{ and } C \text{ have the same fixed points and are therefore permutable}\\ &= \{ qC(q) \}, \text{ whatever } q \text{ may be.} \qquad \dots \qquad \dots \qquad \text{(i)}\end{aligned}$$

Hence, assuming the fundamental property of distance, *viz.* $(pq)+(qr)=(pr)$, we have

$$\begin{aligned}\{ pC^n(p) \} &= \{ pC(p) \} + \{ C(p)C^2(p) \} + \dots\dots + \{ C^{n-1}(p)C^n(p) \}\\ &= n\,\{ pC(p) \} \text{ by (i)}\end{aligned}$$

Putting $C^n=C_1$ in this result, we easily prove

$$\{ pC^n(p) \} = n\,\{ pC(p) \} \text{ for all values of } n.$$

If $\{ pC(p) \}$ is the unit of distance, we see that $\{ pC^n(p) \} = n$; hence, *if pq is the unit of distance and C is the unique correspondence of* (C) *which carries p to q, any distance LM must be defined as the number n, where $M=C^n(L)$.*

For a proof on different lines see Whitehead's *Universal Algebra*, Vol. (I), page 353.

Several of the theorems in this paper have analogues in the general Linear Transformation. These I shall discuss in a future paper.

SHORT NOTES.

Note on sn, cn, dn of $(u_1 + u_2 + u_3)$.

Symmetrical results for the expression of sn, cn, dn of $(u_1+u_2+u_3)$ in terms of those of u_1, u_2, u_3 given by Dr. Glaisher and Prof. Cayley are quoted in Whittaker's *Modern Analysis*, 2nd Edn, Chap. XXII, Misc. Ex., 19 and 20.

The following additional symmetrical forms may be of interest.

(1)

$$\operatorname{sn}(u_1+u_2+u_3)=\frac{X}{W},$$

$$\operatorname{cn}(u_1+u_2+u_3)=\frac{Y}{W},$$

$$\operatorname{dn}(u_1+u_2+u_3)=\frac{X}{W};$$

where

$$X=\sum\left[\frac{\partial s_3}{\partial u_3}.\, s_1 s_2(s_1^2-s_2^2)\right]$$

$$Y=-\sum\left[\frac{\partial c_3}{\partial u}.\, c_1 c_2(c_1^2-c_2^2)\right]$$

$$Z=-\frac{1}{k^4}\sum\left[\frac{\partial d_3}{\partial u_3}.\, d_1 d_2(d_1^2-d_2^2)\right]$$

$$\begin{aligned}W&=-\sum\left[s_1.\frac{\partial s_1}{\partial u_1}(s_2^2-s_3^2)\right]\\&=\sum\left[s_1.\frac{\partial s_1}{\partial u_1}(c_2^2-c_3^2)\right]\\&=\frac{1}{k^2}\sum\left[s_1\frac{\partial s_1}{\partial u_1}(d_2^2-d_3^2)\right]\\&=\sum\left[s_1 s_2\left(s_1.\frac{\partial s_2}{\partial u_2}-s_2.\frac{\partial s_1}{\partial u_1}\right)\right]\\&=\sum\left[c_1 c_2\left(c_1.\frac{\partial c_2}{\partial u_2}-c_2\frac{\partial c_1}{\partial u_1}\right)\right]\\&=\frac{1}{k^4}\sum\left[d_1 d_2\left(d_1\frac{\partial d_2}{\partial u_2}-d_2.\frac{\partial d_1}{\partial u_1}\right)\right],\end{aligned}$$

the summation extending to the suffixes 1, 2, 3.

Proof. If $u_1+u_2+u_3+u_4=0$ we can deduce from Jacobi's Fundamental Theta-function formulae that

$$[1133]-[3311]+[4422]-[2244]=0$$

where $\qquad$ [1133] means $\mathrm{I}_1(u_1).\ \mathrm{I}_1(u_2).\ \mathrm{I}_3(u_3).\ \mathrm{I}_3(u_4)$; etc.
the notation I employed for the Theta function corresponding to that of Tannery and Molk followed by Whittaker and Watson in their *Modern Analysis.*

Hence $\qquad s_1\, s_2\, d_3.d_4 - d_1\, d_2\, s_3.s_4 + c_3.c_4 - c_1\, c_2 = 0,$

or $\qquad d_1\, d_2\, s_3.s_4 - c_3.c_4 - s_1\, s_2\, d_3.d_4 + c_2\, c_3 = 0. \qquad \ldots \quad$ (i)

Similarly $\qquad d_2\, d_3\, s_1.s_4 - c_1.c_4 - s_2\, s_3\, d_1.d_4 + c_2\, c_3 = 0, \qquad \ldots \quad$ (ii)

$\qquad d_3\, d_1\, s_2.s_4 - c_2.c_4 - s_3\, s_1\, d_2.d_4 + c_3\, c_1 = 0. \qquad \ldots \quad$ (iii)

Hence, $\qquad$ since $\mathrm{sn}\,(u_1+u_2+u_3) = -\mathrm{sn}\, u_4 = -s_4$

$$\mathrm{cn}\,(u_1+u_2+u_3) = \mathrm{cn}\, u_4 = c_4.$$
$$\mathrm{dn}\,(u_1+u_2+u_3) = \mathrm{dn}\, u_4 = d_4.$$

We have from (i), (ii), (iii),

$$\frac{\mathrm{sn}(u_1+u_2+u_3)}{\begin{vmatrix} c_3, & s_1\, s_2\, d_3, & c_1\, c_2 \\ c_1, & s_2\, s_3\, d_1, & c_2\, c_3 \\ c_2, & s_3\, s_1\, d_2, & c_3\, c_1 \end{vmatrix}} = \frac{-\mathrm{cn}(u_1+u_2+u_3)}{\begin{vmatrix} d_1\, d_2\, s_3, & s_1\, s_2, d_3, & c_1\, c_2 \\ d_2\, d_3\, s_1, & s_2\, s_3\, d_1, & c_2\, c_3 \\ d_3\, d_1\, s_2, & s_3\, s_1\, d_2, & c_3\, c_1 \end{vmatrix}}$$

$$= \frac{\mathrm{dn}\,(u_1+u_2+u_3)}{\begin{vmatrix} d_1\, d_2\, s_3, & c_3, & c_1\, c_2 \\ d_2\, d_3\, s_1, & c_1, & c_2\, c_3 \\ d_3\, d_1\, s_2, & c_2, & c_3\, c_1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} d_1\, d_2\, s_3, & c_3, & s_1\, s_2\, d_3 \\ d_2\, d_3\, s_1, & c_1, & s_2\, s_3\, d_1 \\ d_3\, d_1\, s_2, & c_2, & s_3\, s_1\, d_2 \end{vmatrix}}$$

Hence we obtain the results written down above.

(2) $$\mathrm{sn}(u_1+u_2+u_3) = \frac{\mathrm{L}}{\mathrm{R}},$$

$$\mathrm{cn}(u_1+u_2+u_3) = \frac{\mathrm{M}}{\mathrm{R}},$$

$$\mathrm{dn}(u_1+u_2+u_3) = \frac{\mathrm{N}}{\mathrm{R}};$$

where

$$\begin{aligned} \mathrm{L} &= \Sigma s_1{}^2 c_2 d_2 c_3 d_3 - s_1 s_2 s_3 [c_2{}^2 d_1{}^2 + c_1{}^2\, d_3{}^2 + c_3{}^2 d_2{}^2] \\ &= \Sigma s_1 c_2 d_2 c_3 d_3 - c_1 c_2 c_3 \Sigma d_1 d_2 s_3 c_3 \\ &\qquad - s_1 s_2 s_3 [3 - (1+k)^2 \Sigma s_1{}^2 + k^2 \Sigma s_1{}^2 s_3{}^2] \\ \mathrm{M} &= c_1 c_2 c_3 [s_1{}^2 d_2{}^2 + s_2{}^2 d_3{}^2 + s_3{}^2 d_1{}^2] - \Sigma c_1{}^2 s_2 d_2 s_3 d_3 \\ &= c_1 c_2 c_3 [\Sigma s_1{}^2 - k^2 \Sigma s_1{}^2 s_2{}^2] - \Sigma c_1 s_2 d_2 s_3 d_3 + s_1 s_2 s_3 \Sigma d_1 d_2 s_3 c_3 ; \end{aligned}$$

$$N = d_1d_2d_3[c_1^2s_1^2 + c_2^2s_3^2 + c_3^2s_1^2] - \Sigma d_1^2c_2s_2c_3s_3.$$
$$= d_1d_2d_3[\Sigma s_1^2 - \Sigma s_1^2s_2^2] - \Sigma d_1c_2s_2c_3s_3 + k^2s_1s_2s_3\Sigma s_1d_1c_2c_3$$
$$R = s_1^2c_2^2d_3^2 + s_2^2c_3^2d_1^2 + s_3^2c_1^2d_2^2 - \Sigma s_1c_1d_1s_2c_2d_2$$
$$= \Sigma s_1^2 - (1+k^2)\Sigma s_1^2s_2^2 + 3k^2s_1^2s_2^2s_3^2 - \Sigma s_1c_1d_1s_2c_2d_2\,;$$

the summation referring to the suffixes 1, 2, 3.

Proof. Suppose $u_1 + u_2 + u_3 + u_4 = 0$,

Then either from Jacobi's Fundamental Theta-function formulae, or from the relations

$$\frac{s_1c_4 + s_4c_1}{d_1 + d_4} + \frac{s_3c_2 + s_2c_3}{d_3 + d_2} = 0$$

and
$$\frac{s_1c_4 - s_4c_1}{d_1 - d_4} + \frac{s_3c_2 - s_2c_3}{d_3 - d_2} = 0.$$

we obtain
$$c_1d_2.s_4 + s_1d_3.c_4 + s_2c_3.d_4 + s_3c_2d_1 = 0\,;$$
similarly
$$c_2d_3.s_4 + s_2d_1.c_4 + s_3c_1.d_4 + s_1c_3d_2 = 0,$$
$$c_3d_1.s_4 + s_3d_2.c_4 + s_1c_2.d_4 + s_2c_1d_3 = 0.$$

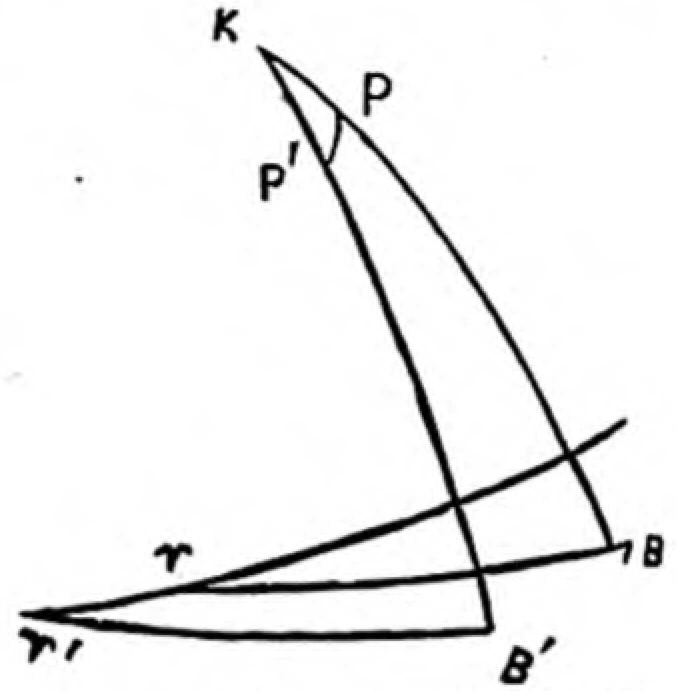
Solving the last three equations for s_4, c_4, d_4 we obtain the results stated.

F. H. V. Gulasekaram.

Precession and Nutation.

The following method of finding the charges in the right ascension and declination of a star on account of precession and nutation may be of interest to those preparing for an examination.

Let P♈B be a quadrantal triangle on the celestial sphere, so that B has a right ascension, 90° P being the pole of the equator. Let P′, ♈′, B′ denote their positions as altered by precession and nutation.



If α, δ be the co-ordinates of a star S then the direction cosines of OS are cos S♈, cos SB, cosSP ; *i.e.*, $\cos\alpha \cos\delta$, $\sin\alpha\cos\delta$, $\sin\delta$ respectively.

We know that θ, the angle between two vectors (l, m, n) and (l', m', n') is $\cos^{-1}(ll'+mm'+nn')$.

Also $\Upsilon'KP'=\dfrac{\pi}{2}=\Upsilon KP$. $\therefore\ \Upsilon'\Upsilon=P'KP$; $KP=\omega$; $KP'=\omega'$.

(i) *Now* $\sin\delta'=\cos SP'=\sin\delta.\cos PP'+\sin\alpha\cos\delta.\cos BP'$
$+\cos\alpha\cos\delta.\cos\delta.\cos\Upsilon P'$
$=\sin\delta(\cos\omega.\cos\omega'+\sin\omega.\sin\omega'\cos k)$
$+\sin\alpha\cos\delta(-\cos\omega'\sin\omega+\sin\omega'\cos\omega\cos k)$
$+\cos\alpha\cos\delta(\sin\omega'.\sin k)$

(ii) $\sin\alpha,\cos\delta'=\cos SB'=\sin\delta\cos PB'+\sin\alpha\cos\delta\cos BB'$
$+\cos\alpha\cos\delta\cos\Upsilon B'$
$=\sin\delta(-\cos\omega\sin\omega'+\sin\omega.\cos\omega'\cos k)$
$+\sin\alpha\cos\delta(\sin\omega.\sin\omega'+\cos\omega\cos\omega'\cos k)$
$+\cos\delta\cos\delta(\cos\omega'.\sin k)$

(iii) $\cos\alpha'\cos\delta'=\cos S\Upsilon'=\sin\delta.\cos P\Upsilon'+\sin\alpha\cos\delta$
$B\Upsilon'+\cos\alpha\cos\phi\cos P\Upsilon B$
$=\sin\delta(-\sin\omega.\sin k)$
$+\sin\alpha\cos\delta(\sin k\cos\omega)+\cos\alpha\cos\delta\cos.k.$

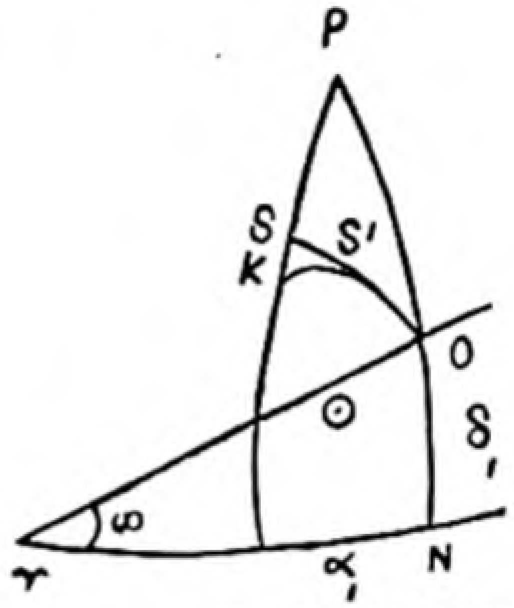
The rest of the working is the same as that given in Ball's. *Spherical Astronomy*, p. 189.

K. K. Ranganatha Aiyar.

Annual Parallax.

The following simple geometrical method may be substituted for Art iii, p. of 330 Ball's *Spherical Astronomy* to find the effect of parallax on the right ascension and declination of a fixed star.

The effect of parallax is to throw the star from its mean place towards the sun through an angle $=\sigma \sin E$ where σ is the amount of annual parallax and E is the angular distance between the star and the sun.



Let P be the pole of the equator, S, S′ the true and apparent positions of the star.

Then $SS' = \sigma \sin SO$, where O is the sun.

If (α, δ) be the R.A., and declination of the star, then

$$(1)\quad \alpha' - \alpha = S'K \sec. \delta$$
$$= \sigma \sin SO, \sin S'SK \sec \delta$$
$$= \sigma \sec \delta \sin (\alpha_1 - \alpha) \cos \delta_1$$
$$[\alpha_1, \delta_1 \text{ being the coords, of O}].$$
$$= \sigma \sec \delta \{ \sin \alpha_1 \cos \delta_1 \cos \alpha - \cos \alpha_1 \cos \delta_1 \sin \alpha \}$$
$$= \sigma \sec \delta \{ \cos \alpha \sin \odot \cos \omega - \cos \odot \sin \alpha \},$$

since $\sin \alpha_1 \cos \delta_1 = \sin \alpha_1 \cos \odot \sec \alpha_1 = \cos \omega \sin \odot$,

and $\cos \alpha_1 \cos \delta_1 = \cos \odot$.

$$(2)\quad \delta' - \delta = -SK = SS' \cos S'SP$$
$$= \sigma \sin SO \cos OSP$$
$$= \sigma (\cos OP - \cos OS \cos SP) \operatorname{cosec} SP$$
$$= \sigma (\sin \delta_1 - \cos OS . \sin \delta) \sec \delta$$
$$= \sigma [\sin\delta_1 \cos \delta - \cos \delta_1 \sin \delta \, , \cos (\alpha_1 - \alpha)]$$
from the $\triangle$SPO.
$$= \sigma \{ \sin \odot \sin \omega \cos \delta - \sin \delta \cos \alpha \cos \odot - \sin \delta \sin \alpha \cos \omega \sin \odot \}$$

since $\sin \delta_1 = \sin \odot \sin \omega$

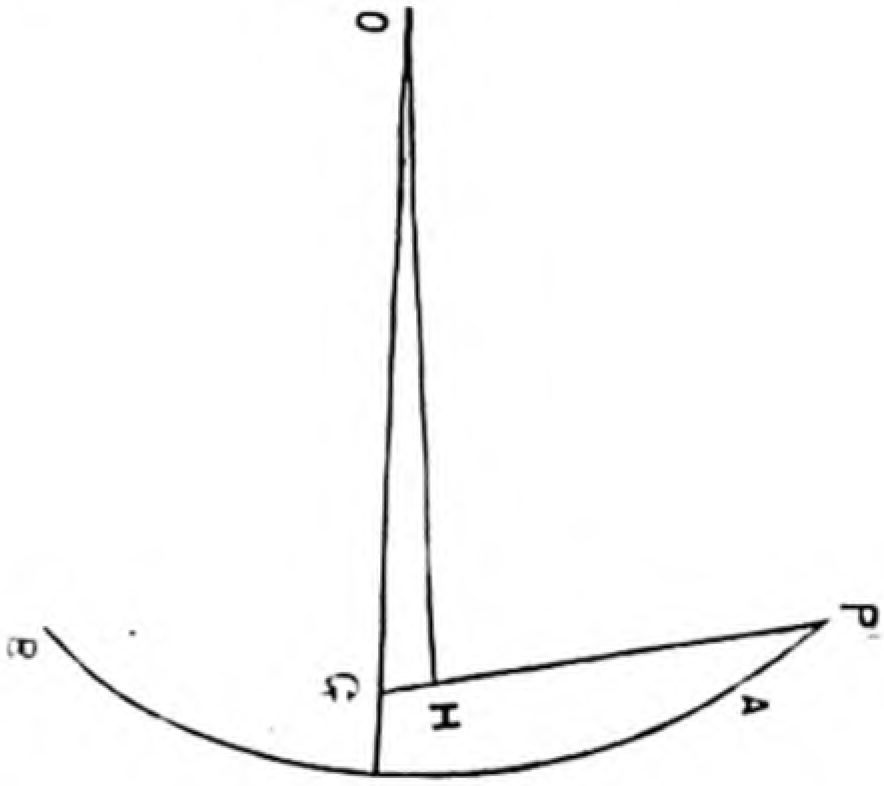
$\cos \delta_1 \cos \alpha_1 = \cos \odot$

$\cos \delta_1 \sin \alpha_1 = \sin \odot \cos \omega$; from (1).

C. Krishnamachary,

Centroid of a Uniform Circular Arc

Let n equal particles of unit mass be placed at equal intervals α along the arc AB whose centre is O, so that $A\hat{O}B=(n-1)\alpha$; and let x = distance of the centroid G from O. Add an equal particle at P where $AP=\alpha$. Then the centroid of the $(n+1)$ particles is in PG and must be H such that $H\hat{O}G=\frac{1}{2}\alpha$, since, by symmetry, OG, OH are the bisectors of the angles AOB, POB respectively.



$$\therefore \qquad \tan\tfrac{1}{2}\alpha=\frac{a}{a\cos\left[\frac{1}{2}(n-1)+1\right]\alpha+nx},$$

by the usual formula for 'centre of mass'.

Hence $nx\sin\frac{1}{2}\alpha=a\sin\left[\left(\frac{1}{2}n+\frac{1}{2}\right)\alpha-\frac{1}{2}\alpha\right]$

$$=a\sin\tfrac{1}{2}n\alpha.$$

$$\therefore\ n=a\sin\left(\tfrac{1}{2}n\alpha\right)/\left(n\sin\tfrac{1}{2}\alpha\right).$$

Cor. When n is infinite, since $(n-1)\alpha=A\hat{O}B=2\theta$, $\frac{1}{2}n\alpha=\theta$; that is, the centroid of the uniform circular arc AB is given by

$$x=a(\sin\theta)/\theta.$$

M. T. Naraniengar.

SOLUTIONS.

Question 649.

(S. MALHARI RAO, B.A.) :—A gentleman has one-third of an acre of vacant land round his house. He wishes to divide it into five different plots to grow five different flowers. What must be the areas of these plots so that they may be in A. P., and each of them may be contained an exact number of times in an acre? Shew that there cannot be more than one set of values for the areas.

Remarks by Eric H. Neville.

The solution is much abbreviated if we take into account from the start the condition

$$\frac{1}{y}-\frac{1}{x}<\frac{1}{15}$$

which in the solution on page 60 of the *Journal* is used only at the end. This condition, in virtue of the equation

$$\frac{1}{y}+\frac{1}{x}=\frac{2}{15},$$

is equivalent to $x<3y$,

and we have also $y<x$.

(i) If a is an integer and

$$x=15a=(2a-1)y,$$

then $2a-1$ is an odd integer between 1 and 3, which is impossible.

(ii) If a, b are integers, and

$$x=3a,\ y=5b,\ 3a+5b=2ab,$$

then since

$$5b/a=2b-3,$$

$5b/a$ is an odd integer which being equal to $3y/x$ is less than 3 and greater than 1, which again is impossible.

(iii) If a is an integer and

$$y=15a=(2a-1)\,x,$$

then $2a-1$ is an integer less than 1.

(iv) If a, b are integers and

$$x=5a,\ y=3b,\ 5a+3b=2ab,$$

then since

$$3b/a=2b-5,$$

$3b/a$ is an odd integer, which being equal to $5y/x$ is less than 5 and greater than $5/3$, and therefore is necessarily 3. Thus the *sole* solution fulfilling the conditions assigned, is given by

$$a=b=4,\ x=20,\ y=12.$$

Question 691.

(A. NARSINGA RAO) :—A marble slab of n pound breaks into k pieces with which a tradesman finds himself able to weigh goods from 1 to n pounds (fraction excluded). Shew that the least value of λ is the smallest integer satisfying the relation $3^k \geq 2n+1$.

What are the weights of the several pieces?

Additional solution by T. P. Trivedi, M.A., LL.B.

It is obvious that when weighing, each piece can be dealt with in 3 ways. It may be put in the first scale pan, or in the second, or it may not be used at all; thus the k pieces can be dealt with in 3^k ways; excluding the case when no piece is used, the number of weighings is 3^k-1; but since the weights in the two pans can be interchanged, the number of *distinct* weighings possible is $\frac{3^k-1}{2}$ and this must be not$<n$.

$$\therefore \qquad \frac{3^k-1}{2} \not< n, \text{ or } 3^k \not< 2n+1.$$

Again, since there are 3 ways of dealing with the weights, it is obvious that the weights will form a G. P. of which 3 is the common ratio. Hence the weights will be 1, 3, 9, 27 etc.

Taking 4 weights (1, 3, 9, 27), the number of *distinct* weighings is $\frac{3^4-1}{2}+40$; and since the maximum that can be weighed is $1+3+9+27$ $=40$, and the minimum is 1, all the weighings from 1 to 40 pounds are possible in one and one way only.

Question 739.

(S. RAMANUJAN) :—Show that

$$\int_0^\infty e^{-nx}(\cot x+\coth x)\sin nx\,dx=\frac{\pi}{2}\left(\frac{1+e^{-n\pi}}{1-e^{-n\pi}}\right)^{(-1)^n}$$

for all positive integral values of n.

Solution by A. C. L. Wilkinson.

Consider the integral $\int e^{-n(1-i)z}\cot z\,dz$ over the following contour :—

The real axis of x indented with semi-circular contours of radii $\rho\to 0$ surrounding the points $\pi, 2\pi,\ldots$; an infinitely great quarter

circle, the imaginary axis, and a quarter circle of radius ρ surrounding the origin. By Cauchy's theorem this is zero.

For the infinitely great quarter circle the integral becomes comparable with

$$\int_0^{\frac{1}{2}\pi} e^{-n\mathrm{R}(\cos\Theta+\sin\Theta)}\,\mathrm{R}\,d\Theta,$$

and this part is zero.

We thus get

$$\int_{\rho}^{\pi-\rho}+\int_{\pi+\rho}^{2\pi-\rho}+\int_{2\pi+\rho}^{3\pi-\rho}+\ldots e^{-nx}(\cos n\,x+i\sin nx)\cot x\,dx$$

$$-\int_0^{\infty} e^{-nx}(\cos nx-i\sin\ nx)\coth x\,dx+\underset{\rho\to 0}{\mathrm{Lt}}\int_{\frac{1}{2}\pi}^{0} i\,d\Theta$$

$$+\sum_{r=1}^{r=\infty}\underset{\rho\to 0}{\mathrm{Lt}}\int_{\pi}^{0} e^{-n(1-i))\,r\pi}\,i\,d\Theta=0.$$

Equate the imaginary part and since $\int e^{-nx}\sin n\,x\cot x\,dx$ is no longer infinite at $x=r\pi$, we can write

$$\int_0^{\infty} e^{-nx}(\cot x+\coth x)\sin n\,x\,dx$$

$$=\frac{\pi}{2}\left(1+2\Sigma e^{-nr\pi}\cos nr\pi\right)$$

$$=\frac{\pi}{2}\left(\frac{1+e^{-n\pi}}{1-e^{-n\pi}}\right)^{(-1)^n}.$$

Question 740.

(S. RAMANUJAN):—If

$$\phi(x)=\frac{e^x\{[x]^2\}!}{[x]}-2\pi x$$

where $[x]$ denotes the greatest integer in x, show that $\phi(x)$ is a continuous function of x for all positive values of x, and oscillates from $\frac{\pi}{3}$ to $-\frac{\pi}{6}$ when x becomes infinite. Also differentiate $\phi(x)$.

Solution by A. C. L. Wilkinson

The function is continuous at $x=r$, for

$$\underset{h\to o}{\text{Lt.}}\left\{\frac{er-1)\,(!^{-h})}{(r-h)^{r-1}}\right\}^2-2\pi(r-h);\ \left\{\frac{e^r.r!}{r^r}\right\}^2-2\pi r;$$

and $$\underset{h\to o}{\text{Lt.}}\left\{\frac{e^{r+h}.r!}{(r+h)^r}\right\}^2-2\pi(r+h)\text{ are the same}$$

The progressive and regressive differential coefficients at these points are however different.

Consider $x=n$ when n is large

$$\phi(n)=\left\{\frac{e^n(n-1)!}{n^{n-1}}\right\}^2-2\pi n=2\pi n\left\{1+\frac{1}{6n}+\cdots\right\}-2\pi n$$

$$=\frac{\pi}{3},\text{ by use of Sterling's theorem;}$$

and so for $x=n\pm\epsilon$, since (x) is continuous, $\phi(n\pm\epsilon)=\frac{\pi}{3}$.

$$\text{Consider }\frac{d\phi(x)}{dx}=2\,\frac{e^x([x])!}{x^{[x]}}\cdot\left\{\frac{e^x[x]}{x^{[x]}}-\frac{e^x.[x].([x])!}{x^{[x]+1}}\right\}-2\pi$$

$$=2\left\{\frac{e^x([x])!}{x^{[x]}}\right\}^2\left(1-\frac{[x]}{x}\right)-2\pi.$$

Now when n is very large $1-\frac{[x]}{x}\to 0$, but $x-[x]$ can have any value between 0 and 1.

Thus $\frac{d\phi}{dx}=2(2\pi x+k)\left\{\frac{x-[x]}{x}\right\}-2\pi=4\pi(x-[x])-2\pi$, in the limit and $\frac{d\phi}{dx}=0$ for $x=[x]+\frac{1}{2}$; and this gives the minimum value of $\phi(x)$ when x is large.

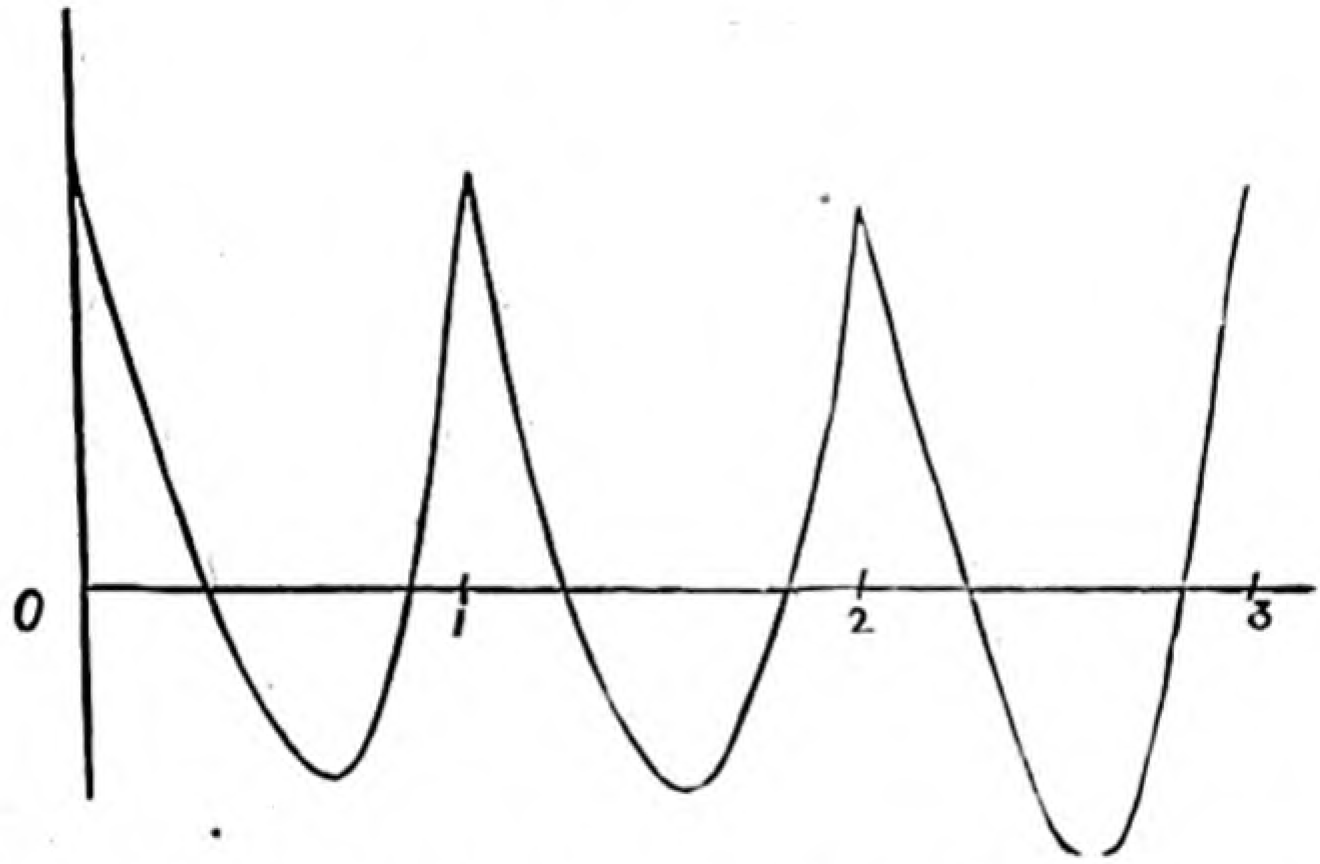
We evalute therefore

$$\left\{\frac{e^{n+\frac{1}{2}}.n\,!}{(n+\frac{1}{2})^n}\right\}^2-2\pi(n+\tfrac{1}{2})$$

$$=\frac{e.n^{2n}}{(n+\frac{1}{2})^{2n}}.2\pi n\left(1+\frac{1}{6n}+\dots\right)-2\pi\left(n+\frac{1}{2}\right)$$

$$=2\pi n\left(1+\frac{1}{4n}+\dots\right)\left(1+\frac{1}{6n}+\right)-2\pi\left(n+\frac{1}{2}\right)$$

$$=-\frac{\pi}{6}.$$



[A graph of the function between $x=0$ and $x=3$ is attached.]

Question 746.

(LAKSHMI SHANKAR N. BHATT) :—Prove that, n being an integer not less than 2,

$$\left\{\frac{1}{n-1}+\frac{1}{2}\cdot\frac{1}{3n-1}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5n-1}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7n-1}\dots\text{ad. inf.}\right\}\times$$
$$\left\{\frac{1}{n+1}+\frac{1}{2}\cdot\frac{1}{3n+1}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5n+1}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7n+1}\dots\text{ad. inf.}\right\}$$
$$=\frac{2}{(n-1)(n+1)}+\frac{2}{(3n-1)(3n+1)}+\frac{2}{(5n-1)(5n+1)}\dots$$

Solution by S. V. Venkatachalu Iyer and K. Appukuttan Erady.

Now since

$$\frac{x^{n-1}}{n-1}+\frac{1}{2}\frac{x^{3n-1}}{3n-1}+\frac{1\cdot3}{2\cdot4}\frac{x^{5n-1}}{5n-1}\dots\dots\text{ad. inf.}$$
$$=\int x^{n-2}(1-x^{2n})^{-\frac{1}{2}}dx,$$

and
$$\frac{x^{n+1}}{n+1}+\frac{1}{2}\cdot\frac{x^{3n+1}}{3n+1}+\frac{1\cdot 3}{2\cdot 4}\,\frac{x^{5n+1}}{5n+1}+\ldots\ldots\text{ad. inf.}$$
$$=\int x^{n}(1-x^{2n})^{-\frac{1}{2}}\,dx,$$

and since both the series are convergent for values of x lying between the limits 0 and 1 (inclusive), we have, for integral values of $n \not< 2$, the left hand side of the given identity

$$=\int_0^1 x^{n-2}(1-x^{2n})^{-\frac{1}{2}}dx\times\int_0^1 x^{n}(1-x^{2n})^{-\frac{1}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{n-2}{n}}.\ \sec\theta.\ \frac{1}{n}(\sin\theta)^{\frac{1}{n}-1}\cos\theta\,d\theta$$

$$\times\int_0^{\frac{\pi}{2}}\sin\theta.\ \sec\theta.\ \frac{1}{n}(\sin\theta)^{\frac{1}{n}-1}\cos\theta\,d\theta$$

[by the substitution $x=\sin\theta$]

$$=\frac{1}{n^2}\cdot\int_0^{\frac{\pi}{2}}(\sin\theta)^{-\frac{1}{n}}\,d\,\theta.\int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{1}{n}}\,d\theta$$

$$=\frac{\Gamma\left(\frac{n-1}{2n}\right).\Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{2n-1}{2n}\right).\Gamma\left(\frac{2n+1}{2n}\right)}=\frac{\pi}{2n}\tan\frac{\pi}{2n}, \qquad \ldots \quad (1)$$

since
$$\Gamma n\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}.$$

But since $\cos\theta=\left\{1-\frac{4\theta^2}{\pi^2}\right\}\left\{1-\frac{4\theta^2}{3^2\pi^2}\right\}\ldots\ldots,$

$$\cos\frac{\theta}{2}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\left(1-\frac{\theta^2}{5^2\pi^2}\right)\ldots\ldots$$
$$=\frac{\pi-\theta}{\pi}\cdot\frac{\pi+\theta}{\pi}\cdot\frac{3\pi-\theta}{3\pi}\cdot\frac{3\pi+\theta}{3\pi}\ldots\ldots$$

Taking logarithms and differentiating

$$\tfrac{1}{2}\tan\frac{\theta}{2}=\left[\frac{\pi}{\pi-\theta}\cdot\frac{1}{\pi}-\frac{\pi}{\pi+\theta}\cdot\frac{1}{\pi}\right]$$
$$+\left[\frac{3\pi}{3\pi-\theta}\cdot\frac{1}{3\pi}-\frac{3\pi}{3\pi+1}\cdot\frac{1}{3\pi}\right]\ldots$$
$$=\left\{\frac{1}{\pi-\theta}-\frac{1}{\pi+\theta}\right\}+\left\{\frac{1}{3\pi-\theta}-\frac{1}{3\pi+\theta}\right\}\ldots$$
$$=\frac{2\theta}{(\pi-\theta)(\pi+\theta)}+\frac{2\theta}{(3\pi-\theta)(3\pi+\theta)}\ldots\ldots$$

Putting $\theta=\frac{\pi}{n}$, we have

$$\tfrac{1}{2}\tan\frac{\pi}{2n}=\frac{\frac{2\pi}{n}}{\left(\pi-\frac{\pi}{n}\right)\left(\pi+\frac{\pi}{n}\right)}+\frac{\frac{2\pi}{n}}{\left(3\pi-\frac{\pi}{n}\right)\left(3\pi+\frac{\pi}{n}\right)+}\ldots\ldots(2)$$

$$\therefore\ \frac{\pi}{2n}\tan\frac{\pi}{2n}=\frac{2}{(n-1)(n+1)}+\frac{2}{(3n-1)(3n+1)}+\ldots\ldots$$

From (1) and (2), the given result follows.

Question 749.

(S. Krishnaswami Aiyangar) :—If ρ be the radius of curvature of the curve $r^m=a^m\sin m\theta$, ρ_k the radius of curvature at the corresponding point of its k^{th} negative pedal with regard to the pole, show that $(1-mk+m)^{m-1}a^{mk}\rho_k{}^{m-1}=(1-mk)^{m-1}\{(m+1)\rho\}^{mk+m-1}$.

Solution by S. V. Venkataraya Sastri, M.A., L.T., M. M. Thomas and others.

The k^{th} negative pedal is $r^n=a^n\sin n\theta$, where $n=m/(1-mk)$.

The point on the k^{th} negative pedal corresponding to (r,θ) in the original curve is $[r^{1-mk}a^{mk},\ (mk+k-1)\theta.]$

We know
$$\rho=\frac{a^m}{(m+1)r^{m-1}}$$

$$\therefore\qquad \rho_k=\frac{a^n}{(n+1)r^{n-1}}.$$

Hence, after a slight simplification, the required result follows.

Additional solutions by K. Srinivasan, K. B. Madhava and S. V. Venkatachala Iyer.

Question 760.

(K. Appukuttan Erady, M. A.) :—If

$$f(n)\equiv\frac{1}{n}+\frac{1}{n(n+1)}+\frac{1}{n(n+1)(n+2)}+\cdots,$$

show that

$$f(1)+\frac{f(2)}{1!}+\frac{f(3)}{2!}+\ldots\ldots=e,$$

Solution (1) *by M. K. Kewalramani, M.A., G. V. M. Gaitonde and others;* (2) *by M. Viraswamaiya, Durai Rajan, K. K. Ranganatha Aiyar and Lakshmi Sankar N. Bhatt B.A.* (3) *by A. Narasinga Rao;* *and* (4) *by K. B. Madhava.*

(1) Let $f(1)=t_1:\dfrac{f(2)}{1!}=t_2:\dfrac{f(3)}{2!}=t_3$, etc. and S their sum.

$$t_r=\frac{f(r)}{(r-1)!}$$
$$=\frac{1}{(r-1)!}\left\{\frac{1}{r}+\frac{1}{r(r+1)}+\frac{1}{r(r+1)(r+2)}+\dots\right\}$$
$$=\frac{1}{r!}+\frac{1}{(r+1)!}+\frac{1}{(r+2)!}+\frac{1}{(r+3)!}+\dots$$

$$\therefore\qquad t_1=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots$$
$$t_2=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\dots$$
$$t_3=\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\dots$$
$$t_4=\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\dots$$

From the above it is clear that the sum S contains:

one $\dfrac{1}{1!}$; two $\dfrac{1}{2!}$; three $\dfrac{1}{3!}$; four $\dfrac{4}{4!}$; five $\dfrac{5}{5!}$

and so on.

$$S=\frac{1}{1!}+2\left(\frac{1}{2!}\right)+3\left(\frac{1}{3!}\right)+4\left(\frac{1}{4!}\right)+5\left(\frac{1}{5!}\right)+6\left(\frac{1}{6!}\right)$$
$$+\dots\dots\dots\dots+r\left(\frac{1}{r!}\right)+(r+1)\frac{1}{(r+1)!}+\dots$$
$$=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\dots\dots\dots\dots$$
$$=e.$$

(2) Now
$$nf(n)=1+\frac{1}{(n+1)}+\frac{1}{(n+1)(n+2)}+\dots$$
$$=1+f(n+1)$$
$$\therefore\qquad nf(n)-f(n+1)=1.$$

In this give to n values 1, 2, 3,...and we get a series of identities

$$f(1)-f(2)=1$$
$$2f(2)-f(3)=1$$
$$3f(3)-f(4)=1,\ \textit{etc.}$$

or which is the same thing as

$$f\ (1)-f(2)=1$$
$$2f\ (2)-f(3)=1$$
$$\frac{3}{2!}f\ (3)-\frac{1}{2!}f\ (4)=\frac{1}{2!}$$
$$\frac{4}{3!}\ f\ (4)-\frac{1}{3!}\ f\ (5)=\frac{1}{3!}$$

and so on, adding the various identities, we get

$$f(1)+\frac{f(2)}{1!}+\frac{f(3)}{2!}+\frac{f(4)}{3!}+\frac{f(5)}{1!}+.$$
$$=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1!}{4}+\ldots$$

(3) $$f(n)=n^{(-1)}+n^{(-2)}+n^{(-3)}\ldots\ldots$$
$$=\left(1-\Delta+\frac{\Delta^2}{2!}-\frac{\Delta^3}{3!}\ldots\right)\frac{1}{n}$$
$$=e^{-\Delta}\left(\frac{1}{n}.\right)$$

Hence $$\frac{e}{n}=e^{1+\Delta}.e^{-\Delta}\frac{1}{n}=e^{\mathrm{E}}f(n)$$
$$=f(n)+\frac{f(n+1)}{1!}+\frac{f(n+2)}{2!}+\ldots \qquad \ldots(\text{i})$$

The given result follows on putting $n=1$.

(4) We have Prym's Identity (see p. 149, vol. vii., or p. 17, vol. viii, J . I. M. S.)

$$f(n)\equiv e\left[\frac{1}{n}-\frac{1}{1!}\ \frac{1}{n+1}+\frac{1}{2!}\ \frac{1}{n+2}-\frac{1}{3!}\ \frac{1}{n+3}+\cdots\right]$$

from which by successive substitutions we have

$$f(1)+\frac{f(2)}{1!}+\frac{f(3)}{2!}+\ldots$$
$$=e\left[1-\frac{1}{1!}\frac{1}{2}+\frac{1}{2!}\frac{1}{3}-\frac{1}{3!}\frac{1}{4}+\ldots\right.$$
$$+\frac{1}{1!}\left\{\frac{1}{2}-\frac{1}{1!}\frac{1}{3}+\frac{1}{2!}\ \frac{1}{4}-\ldots\right.$$
$$\left.+\frac{1}{2!}\left\{\frac{1}{3}-\frac{1}{1!}\frac{1}{4}+\ldots\right.\right]$$
$$=e,$$

since $\frac{1}{n!}-\frac{1}{1!}\frac{1}{n-1!}+\frac{1}{2!}\ \frac{1}{n-2!}-\ldots$to $(n+1)$ terms

$$=\text{coefficient of } x^n \text{ in } e^x\times e^{-x}=0.$$

Question 767.

(S. Krishnaswami Iyengar) :—Show that

$$1-\frac{1}{2^2.5}+\frac{1}{2^4.9}\ldots(-)^n\frac{1}{2^{2n}(4n+1)}=\tfrac{1}{4}\log 5+\tfrac{1}{2}\tan^{-1}2.$$

Solution by Prof. K. J. Sanjana and K. B. Madhava.

More generally,

let $$f(x)=x-\frac{1}{2^2}\frac{x^5}{5}+\frac{1}{2^4}\frac{x^9}{9}-\ldots\ ;$$

then $$f'(x)=1-\frac{1}{2^2}x^4+\frac{1}{2^4}x^8-\ldots=\frac{4}{4+x^4}.$$

Hence $f(x)=C+\int\frac{4dx}{4+x^4}$.

Now $C=0$ evidently, as $x \not> 1$;

and $$\int\frac{4dx}{4+x^4}=\sqrt{2}\int\frac{dy}{1+y^4}, \text{ putting } x=y\sqrt{2},$$

$$=\frac{\sqrt{2}}{4}\sqrt{2}\left\{\tanh^{-1}\frac{y\sqrt{2}}{1+y^2}+\tan^{-1}\frac{y\sqrt{2}}{1-y^2}\right\}$$

$$=\tfrac{1}{2}\left(\tanh^{-1}\frac{2x}{2+x^2}+\tan^{-1}\frac{2x}{2-x^2}\right).$$

Putting $x=1$, we get

the given series $=\tfrac{1}{2}(\tanh^{-1}\tfrac{2}{3}+\tan^{-1}2)$

$=\tfrac{1}{4}\log 5+\tfrac{1}{2}\tan^{-1}2.$

Similar solutions by M. K. Kewalramani, S. V. Venkatachala Aiyar and K. Appukuttan Erady.

Question 768.

(S. Ramanujan) :—If $\psi(x)=\dfrac{x+2}{x^2+x+1}$, show that

(i) $\tfrac{1}{3}\psi(x^{\frac{1}{3}})+\tfrac{1}{9}\psi(x^{\frac{1}{9}})+\tfrac{1}{27}\psi(x^{\frac{1}{27}})+\ldots=\dfrac{1}{\log x}+\dfrac{1}{1-x}$

for all positive values of x ; and

(ii) $\tfrac{1}{3}\psi(x^{\frac{1}{3}})+\tfrac{1}{9}\psi(x^{\frac{1}{9}})+\tfrac{1}{27}\psi(x^{\frac{1}{27}})+\ldots=-\dfrac{1}{1-x}$

for all negative values of x.

Solution by K. J. Sanjana and N. Durairajan.

(i) Since
$$\frac{x+2}{x^2+x+1}=\frac{1}{x-1}-\frac{3}{x^3-1},$$

$$\text{the given series}=\left(\frac{1}{3}\frac{1}{x^{\frac{1}{3}}-1}-\frac{1}{x-1}\right)+\left(\frac{1}{9}\frac{1}{x^{\frac{1}{9}}-1}-\frac{1}{3}\frac{1}{x^{\frac{1}{3}}-1}\right)$$
$$+\left(\frac{1}{27}\cdot\frac{1}{x^{\frac{1}{27}}-1}-\frac{1}{9}\frac{1}{x^{\frac{1}{9}}-1}\right)+\ldots$$
$$=-\frac{1}{x-1}+\frac{1}{3^n}\cdot\frac{1}{x^{3^{-n}}-1}, \text{ for } n \text{ terms.}$$

When $n\to\infty$, the limit of the second part is $\dfrac{1}{\log x}$.

(ii) Put $y=-x$, so that y is positive. Then
$$\psi(x)=\frac{2-y}{1-y+y^2}=\frac{3}{1+y^3}-\frac{1}{1+y}.$$

Proceeding as above we shall get the sum of n terms to be
$$\frac{1}{1+y}-\frac{1}{3^n}\frac{1}{1+y^{3^{-n}}};$$
the limit of the second term for $n\to\infty$ being zero,
$$\text{the required sum}=\frac{1}{1-x}.$$

Similarly, the limit of
$$\sum_1^\infty \frac{1}{r^n}\psi\left(x^{r^{-n}}\right)=\frac{1}{\log x}-\frac{1}{x-1} \text{ or } \frac{-1}{x-1},$$
if $\psi(x)$ stands for $\dfrac{(r-1)+(r-2)+(r-3)x^2+\ldots x^{r-2}}{1+x+x^2+\ldots x^{r-1}}$.

Question 775.

(K. Srinivasan):—Prove geometrically $1+\text{dn}\,2z=\dfrac{2\text{dn}^2z}{1-k^2\text{sn}^4z}$

Solution by F. H. V. Gulasekaram, B.A.,

Take any four points M, m, n N in order on a st. line, and on MN, mn as diameters describe circles. Let the centres be O, O_1; the radii R, r; and let $O\,O_1=\delta$.

Draw chords of the outer circle MA, AD, MB to touch the inner. Draw another circle having its centre on MN and touching AB and MD.

Let the points of contact be as shewn in the figure.

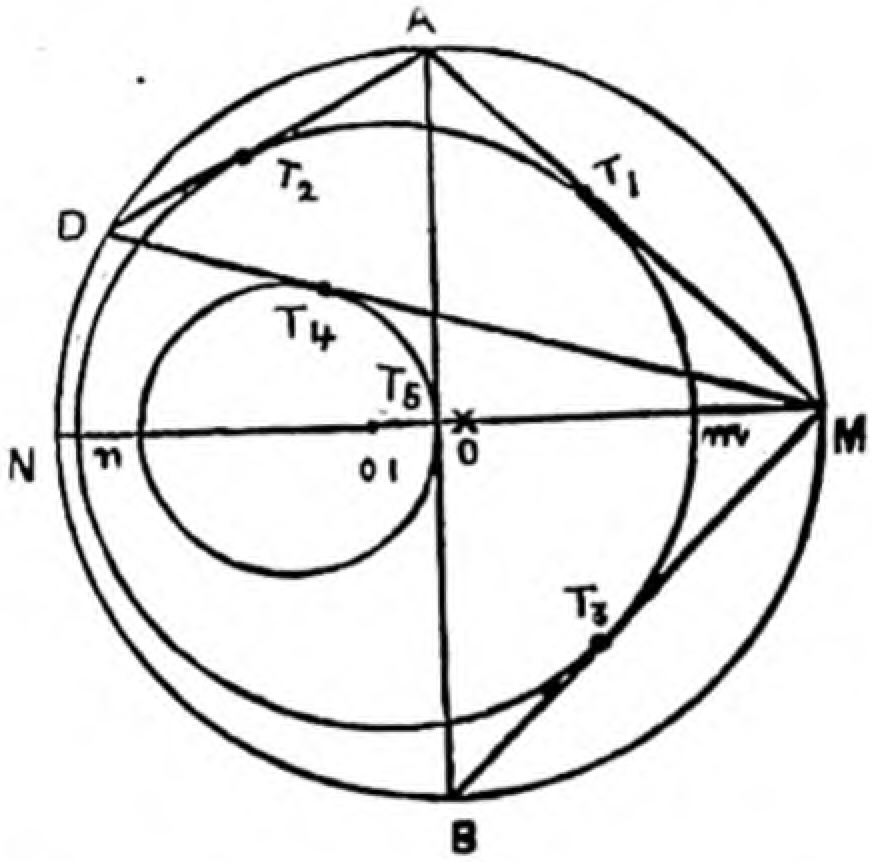
It is evident that T_5 is the mid-point of AB and that it lies on MN.

As in Dixon: *Elliptic Functions*, §§ 105—107, let us suppose that the point M on the outer circle corresponds to the argument zero, A to the argument u; hence D corresponds to the argument $2u$ and B to $-u$.

Also,

$$\mathrm{dn}(u, k)=\frac{AT_1}{MT_1}, \text{ where } k^2=\frac{4R\delta}{(R+\delta)^2-r^2};$$

and
$$M\hat{N}A=\theta=\tfrac{1}{2}\text{ am. } u, \quad M\hat{N}D=\phi=\tfrac{1}{2}\text{ am. } 2u$$



Hence we have

$$\frac{OA^2}{OM^2}=\frac{R^2+\delta^2+2R\delta\cos 2\theta}{OM^2}$$
$$=1-\frac{4R\delta\sin^2\theta}{OM^2}$$
$$=1-\frac{4R\delta}{(R+\delta)^2-r^2}\cdot\frac{MT^2}{OM^2}$$
$$=1-k^2\sin^4\text{am. } u$$
$$=1-k^2\,\mathrm{sn}^4\,u.$$

Again since it can be proved that the three circles in the fig. have the same radical axis [*Vide*: Ex. 45, p. 417. Nixon: *Euc. Revised*], the elliptic functions have the same modulus whether the inner circle be the one with the centre O_1, or the other touching AB, MD.

Again from the circle touching AB, MD, we have

$$1+\mathrm{dn}\,2u=\frac{\mathrm{MD}}{\mathrm{MT}_4}$$

and
$$\mathrm{dn}\,u=\frac{\mathrm{AT}_5}{\mathrm{MT}_4}$$

$$\therefore \qquad \frac{1+\mathrm{dn}\,2u}{\mathrm{dn}\,u}=\frac{\mathrm{MD}}{\mathrm{AT}_5},$$

$$=\frac{2\mathrm{R}\sin\phi}{2\mathrm{R}\sin\theta\cos\theta}$$

$$=\frac{\sin 2\,\mathrm{O}_1\mathrm{AT}_1}{\sin\theta\cos\theta}.$$

$$=\frac{2.\mathrm{OT}_1}{\mathrm{OA}}\cdot\frac{\mathrm{AT}_1}{\mathrm{OA}}\cdot\frac{\mathrm{OM}}{\mathrm{OT}_1}\cdot\frac{\mathrm{OM}}{\mathrm{MT}_1}$$

$$=\frac{2.\mathrm{AT}_1}{\mathrm{MT}_1}\cdot\frac{\mathrm{OM}^2}{\mathrm{OA}^2}.$$

$$=\frac{2\,\mathrm{dn}\,u}{1-k^2\mathrm{sn}^4u}$$

Hence
$$1+\mathrm{dn}\,2u=\frac{2\,\mathrm{dn}^2\,u}{1-k^2\,\mathrm{sn}^4u}.$$

Question 776.

(K. Srinivasan) :—Expand in a Fourier series cn^3z.

Solution by F. H. V. Gulasekaram, B.A. and K. B. Madhava.

Method 1.—We have

$$\frac{d}{dz}(\mathrm{sn}\,z\,\mathrm{dn}\,z)=2k^2\mathrm{cn}^3z+\mathrm{cn}\,z\,(1-2k^2)$$

Hence
$$2\,k^2\,\mathrm{cn}^3\,z=-\frac{d^2}{dz^2}\,\mathrm{cn}\,z-\mathrm{cn}\,z\,(1-2\,k^2) \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Now
$$\mathrm{cn}\,z=\frac{2\pi}{\mathrm{K}k}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}\cos(2n+1)\,x}{1+q^{2n+1}},$$

valid throughout the strip $|\mathrm{I}(x)|<\frac{1}{2}\pi\,\mathrm{I}(\mathrm{J})$,

where
$$z=\frac{2\mathrm{K}x}{\pi},\ q=e^{\pi i\mathrm{J}};\ \mathrm{J}=\frac{i\mathrm{K}'}{\mathrm{K}}.$$

(*Vide*: *Ex*. 1, *p*. 504, Whittaker's *Modern Analysis*.)

Hence from (i)

$$\mathrm{cn}^3z=\sum_{u=0}^{\infty}\left[\frac{(2n+1)^2}{2k^2}\left(\frac{\pi}{2\mathrm{K}}\right)^2-\frac{(1-2k^2)}{2k^2}\right]\frac{2\pi q^{n+\frac{1}{2}}\cos(2n+1)\,x}{\mathrm{K}\,(1+q^{2n+1})}$$

valid throughout the strip $|\mathrm{I}(x)|<\frac{1}{2}\pi\,\mathrm{I}\,(\mathrm{J})$.

(*Vide*: Whittaker *Ex*. 57, p. 528, for the analogous expansion of $\mathrm{sn}^3 z$.

Method 2.—If $z=\dfrac{2Kx}{\pi}$, $\mathrm{cn}^3 z$ is an even periodic function of x (with period 2π) which obviously satisfies Dirichlet's conditions for real values of x and therefore

$$\mathrm{cn}^3 z=\frac{1}{2}\,a_0+\sum_{n=1}^{\infty} a_n \cos nx,$$

the expansion being valid for all real values of x.

The co-efficients a_n are given by the formula

$$\pi a_n=\int_{-\pi}^{+\pi} \mathrm{cn}^3 z \exp\,(nix)\,dx.$$

To evaluate this integral consider $\int \mathrm{cn}^3 z.\ \exp\,(nix)\,dx$, taken round the parallelogram whose corners are

$$-\pi,\ \pi,\ \pi\,\mathrm{J},-2\pi+\pi\,\mathrm{J}.$$

From the periodic properties of $\mathrm{cn}^3 z$ and $\exp\,(nix)$, we see that

$$\int_{\pi}^{\pi \mathrm{J}} \text{ cancels } \int_{-2\pi+\pi\mathrm{J}}^{-\pi};$$

again, $-\pi+\frac{1}{2}\pi\mathrm{J}$ and $\frac{1}{2}\,\pi\mathrm{J}$ are the only poles of the integrand (*qua* function of x) inside the contour.

Again, $\mathrm{cn}^3(z+i\mathrm{K}').\ \exp\left\{ni\left(x+\frac{1}{2}\,\pi\mathrm{J}\right)\right\}$

$$=\frac{+i}{k^3}\,\mathrm{ds}^3 z \exp\,(nix)\,q^{\frac{1}{2}n},\ (\text{where } q=e^{\pi i \mathrm{J}})$$

$$=\frac{iq^{\frac{1}{2}n}}{k^3 z^3}\left[1+\tfrac{1}{2}(1-2k^2)z^2+\dots\right]\left[1+nix-\frac{n^2x^2}{2}+\dots\right];$$

so that, the residue of the integrand at the pole $\frac{1}{2}\pi\mathrm{J}$

$$=iq^{\frac{1}{2}n}\left(\frac{\pi}{2\mathrm{K}}\right)\left[\frac{1-2k^2}{2k^3}-\frac{n^2}{2k^3}\left(\frac{\pi}{2\mathrm{K}}\right)^2\right];$$

and the residue at the pole $-\pi+\frac{1}{2}\pi\mathrm{J}$

$$=-i(-1)^n\,q^{\frac{1}{2}n}\left(\frac{\pi}{2\mathrm{K}}\right)\left[\frac{1-2k^2}{2k^3}-\frac{n^2}{2k^3}\left(\frac{\pi}{2\mathrm{K}}\right)^2\right].$$

Hence

$$\left\{\int_{-\pi}^{\pi}-\int_{-2\pi+\pi\mathrm{J}}^{\pi\mathrm{J}}\right\}\mathrm{cn}^3 z.\ \exp\,(nix)\,dx$$

$$=\frac{\pi^2}{\mathrm{K}}\left[1-(-1)^n\right]\left[\frac{n^2}{2k^3}\left(\frac{\pi}{2\mathrm{K}}\right)^2-\frac{1-2k^2}{2k^3}\right]q^{\frac{1}{2}n}.$$

Writing $(x-\pi+\pi)$ J for x in the second integral, we have

$$\{1-(-1)^n q^n\}\int_{-\pi}^{\pi} \text{cn}^2 z.\exp(nix)\,dx$$

$$=\frac{\pi^2}{K}\Big[1-(-1)^n\Big]\Big[\frac{n^2}{2k^2}\Big(\frac{\pi}{2K}\Big)^2-\frac{1-2k^2}{2k^2}\Big]q^{\frac{1}{2}n}$$

When n is *even*, $a_n=0$; but when n is *odd*

$$a_n=\frac{2\pi}{K}\Big[\frac{n^2}{2^2k}\Big(\frac{\pi}{2K}\Big)^2-\frac{1-2k^2}{2k^3}\Big]\frac{q^{\frac{1}{2}n}}{1+q^n};$$

and a_o is easily seen to be zero.

Hence, we get the expansion already obtained by *Method* 1.

Note:—The Fourier Series for $(\text{sn } u)^m$, $(\text{cn } u)^m$, $(\text{dn } u)^m$ and their reciprocals can be found as follows:

We have, when m is any positive integer,(Dixon's *Elliptic Functions* §§ 56, 59.)

(A) $$\Big\{\frac{1}{m-2}\frac{d^2}{du^2}(s^{m-2})=\frac{d}{du}(s^{m-3}\,cd)\Big\}$$
$$=(m-1)k^2s^m-(m-2)(1+k^2)s^{m-2}+(m-3)s^{m-4}$$

(B) $$\Big\{\frac{1}{m-2}\frac{d^2}{du^2}(s^{-m+2})=-\frac{d}{du}(s^{-m+1}\,cd)\Big\}$$
$$=(m-1)s^{-m}-(m-2)(1+k^2)s^{-m+2}+(m-3)k^2s^{-m+4}$$

where s, c, d stand for sn u, cn u, dn u respectively.

Hence $(\text{cn } u)^m$ can be expanded in a Fourier Series by differentiation as follows:—

(1) When m is odd and greater than one, the series for $(\text{sn } u)^m$ is obtained by differentiation from the series for sn u by means of the reduction formula (A).

(2) When m is even and greater than 2, the series for $(\text{sn } u)^m$ is obtained by differentiation from the series for $\text{sn}^2 u$ by means of the reduction formula (A).

(3) When m is odd and greater than 1, the series for $(\text{ns } u)^m$ is obtained by differentiation from the series for ns u by means of the reduction formula (B),

(4) When m is even and greater than 2, the series for $(\text{ns } u)^m$ is obtained by differentiation from the series for $\text{us}^2 u$ by means of the reduction formula (B).

Hence the expansion for $(\mathrm{sn}\, u)^m$ is made to depend on either that of $\mathrm{sn}\, u$ or $\mathrm{sn}^2 u$, and the expansion of $(\mathrm{ns}\, u)^n$ is made to depend on either that of $\mathrm{ns}\, u$ or $\mathrm{ns}^2 u$.

Again since the expansions for $\mathrm{ns}\, u$ and $\mathrm{ns}^2 u$ are deducible from those of $\mathrm{sn}\, u$ and $\mathrm{sn}^2 u$ by writing $u+i\,K'$ for u in the latter, we see that the expansion of $(\mathrm{sn}\, u)^m$ may be made to depend on that of $\mathrm{sn}\, u$ when m is odd and on that of $\mathrm{sn}^2 n$ when m is even, m being positive or negative

By constructing similar reduction formulae for $(\mathrm{cn}\, u)^m$, $(\mathrm{dn}\, u)^m$, *etc.* the Fourier series for $(\mathrm{cn}\, u)^m$, *etc.*, may be obtained.

Question 785.

(S. Ramanujan) :—Show that

$$\sqrt[3]{\{3(\sqrt[3]{a^3+b^3}-a)(\sqrt[3]{a^3+b^3}-b)\}}=\sqrt[3]{\{(a+b)^2\}}-\sqrt[3]{(a^2-ab+b^2)}.$$

This is analogous to

$$\sqrt{\{2(\sqrt{a^2+b^2}-a)(\sqrt{a^2+b^2}-b)\}}=(a+b)-\sqrt{(a^2+b^2)}.$$

Solution by K. K. Ranganatha Aiyar, R. D. Karve, G. A. Kamtekar, L. N. Datta and L. N. Subramanyam.

In the identity

$(a+b-r)^3=(a+b)^3-r^3-3r(a+b)^2+3r^2(a+b)$, put $a^3+b^3=r^3$;

we have

$$\begin{aligned}(a+b-r)^3&=3ab(a+b)-3r(a+b)^2+3r^2(a+b)\\&=3(a+b)(r-a)(r-b)\end{aligned}$$

$$\therefore \qquad \sqrt[3]{3(r-a)(r-b)}=(a+b)^{\frac{2}{3}}-\left(\frac{a^3+b^3}{a+b}\right)^{\frac{1}{3}},$$

which is equivalent to the result required.

Again, in $(a+b-r)^2=2ab-2r(a+b)+2r^2$,

put $r^2=a^2+b^2$, and we get

$$\therefore \qquad \sqrt{2(r-a)(r-b)}=a+b-r=a+b-\sqrt{a^2+b^2}$$

Question 787.

(M. K. Kewalramani, M. A.) :—Prove that, if a be not an integer,

$$\frac{\pi}{2}\frac{f(x+ah)-f(x-ah)}{\sin a\pi}=\frac{f(x+h)-f(x-h)}{1^2-a^2}$$

$$2-\frac{f(x+2h)-f(x+2h)}{2^2-a^2}+3\frac{f(x+3h)-f(x-3h)}{3^2-a^2}-\ldots$$

Solution by V. M. Gaitonde, K. Âppukuttan Erady and K. K. Ranganatha Aiyar.

We can, when y lies between $\pm\pi$, by using the Fourier's Theorem, shew that

$$\sin ay = \frac{2}{\pi}\sin a\pi\left(\frac{\sin y}{1-a^2} - \frac{2\sin 2y}{2^2-a^2} + \frac{3\sin 3y}{3^2-a^2} - \ldots\right),$$

[Williamson: *Int. Calc.*, p. 400, Ex. 10.]

Hence we get

$$\frac{\pi}{2}\cdot\frac{e^{ayi}-e^{-ayi}}{\sin a\pi} = \frac{e^{yi}-e^{-yi}}{1-a^2} - 2\frac{e^{2yi}-e^{-2yi}}{2^2-a^2} + \ldots$$

Putting $e^{yi} \equiv e^{h\frac{d}{dx}}$ and applying each side to the function $f(x)$, we get at once

$$\frac{\pi}{2}\cdot\frac{f(x+ah)-f(x-ah)}{\sin a\pi} = \frac{f(x+h)-f(x-h)}{1^2-a^2}$$
$$-2\frac{f(x+2h)-f(x-2h)}{2^2-a^2} + 3\frac{f(x+3h)-f(x-3h)}{3^2-a^2} - \ldots$$

Addition Solution by L. N. Bhatt, B.A.

Question 788.

(E. H. Neville):—From the point (u,v) can be drawn four normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; four circles are drawn, each through the feet of three of these normals; shew that the sum of the squares of the radii of these circles is given by

$$2a^2b^2\Sigma r^2 = (a^2+b^2)^3 - (a^2-b^2)(a^2u^2-b^2v^2).$$

Solution by F. H. V. Gulasekaram, B.A., K. B. Madhava, V. M. Gaitonde, K. K. Ranganatha Aiyar, and others.

Let $\alpha,\beta,\gamma,\delta$ be the eccentric angles of the feet of the 4 normals.

Then $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta = \frac{2au}{a^2-b^2}$ ... ... (i)

$\sin\alpha+\sin\beta+\sin\gamma+\sin\delta = \frac{-2bv}{a^2-b^2}$ ... ... (ii)

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = 2+\frac{2(a^2u^2-b^2v^2)}{(a^2-b^2)^2}$ ... (iii)

Consider the circle through the points α,β,γ. This meets the circle again at the point $(\pi+\delta)$; so that, after C. Smith's *Conic Sections*, p. 169,

$$\cos(\alpha+\beta+\gamma) = -\cos\delta$$
$$\sin(\alpha+\beta+\gamma) = \sin\delta.$$

If (x_1, y_1) be the centre, r, the radius of the circle

$$x_1=\frac{a^2-b^2}{4a}\Big[\cos\alpha+\cos\beta+\cos\gamma-\cos\delta\Big]$$

$$=\frac{u}{2}-\frac{a^2-b^2}{2a}\cos\delta. \qquad \text{[from (i)]}$$

$$y_1=-\frac{a^2-b^2}{4b}\Big[\sin\alpha+\sin\beta+\sin\gamma-\sin\delta\Big]$$

$$=\frac{v}{2}+\frac{a^2-b^2}{2b}\sin\delta. \qquad \text{[from (ii)]}$$

$$r_1^2=(x_1+a\cos\delta)^2+(y^2{}_1+b\sin\delta)^2$$

$$=\left(\frac{a^2+b^2}{2a}\cos\delta+\frac{u}{2}\right)+\left(\frac{a^2+b^2}{2b}\sin\delta+\frac{v}{2}\right)^2$$

$$=\frac{(a^2+b^2)^2}{4a^2}\Big[\cos\delta+\frac{au}{a^2+b^2}\Big]^2+\frac{(a^2+b)^2}{4b^2}\Big[\sin\delta+\frac{bv}{a^2+b^2}\Big]^2.$$

Hence taking the radii of the four circles in question, and using the results (i), (ii), (iii) above, we have

$$4a^2b^2\Sigma r^2=b^2(a^2+b^2)^2\Big[2+\frac{2(a^2u^2-b^2v^2)}{(a^2-b^2)^2}\Big]$$

$$+a^2(a^2+b^2)^2\Big[2-2\frac{(a^2u^2-b^2v^2)}{(a^2-b^2)^2}\Big]$$

$$+4a^2b^2(u^2+v^2)+4\frac{a^2+b^2}{a^2-b^2}\,a^2b^2(u^2-v^2)$$

$$=2(a^2+b^2)^3-2(a^2-b^2)(a^2u^2-b^2v^2), \text{ after reduction.}$$

Hence the result.

Additional Solution by K. Appukuttan Erady.

Question 789.

(K. J. SANJANA) :—P, Q, R, S are four co-normal points on an ellipse whose centre is C and axes $2a$ and $2b$, O being the point of concurrence of the normals. If x_r, y_r ($r=1, 2, 3, 4$) denote the centres of the circles QRS, PRS, PQS, PQR respectively, prove that

$$\Sigma x=\text{abscissa of O},\ \Sigma y=\text{ordinate of O}$$

and, that each centre lies on an ellipse whose centre is at the mid-point of CO and whose axes are

$$\frac{a^2-b^2}{2a},\ \frac{a^2-b^2}{2b}.\ \text{[Suggested by Q 788].}$$

Solution by F. H. V. Gulasekaram, B.A., S. V. Venkatachala Aiyar, K. B. Madhava and M. M. Thomas.

Let α, β, γ, δ be the eccentric angles of the four points P, Q R, S; Let O be the point $(u, v.)$

Then if (x_4, y_4) be the centre of the circle PQR,

$$x_4 = \frac{u}{2} - \frac{a^2-b^2}{2b}\cos\delta \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$y_4 = \frac{v}{2} + \frac{a^2-b^2}{2b}\sin\delta \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

[*Vide: Solution to Q.* 788.]

$$\therefore \qquad \Sigma x = 2u - \frac{a^2-b^2}{2a}\Sigma\cos\alpha$$

$$= 2u - \frac{a^2-b^2}{2a}.\frac{2au}{a^2-b^2}$$

$$= u, \text{ the abscissa of O.}$$

Similarly, $\Sigma\, y = v$

$$= \text{the ordinate of O.}$$

From (i) and (ii), it is evident that each centre lies on an ellipse whose centre is at the middle point of CO and whose axes are

$$\frac{a^2-b^2}{2a}, \frac{a^2-b^2}{2b}.$$

Question 794.

(Martyn M. Thomas, M.A.) :—The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ moves in such a manner that the centroid of the triangle ABC always lies on the surface $x^{-2}+y^{-2}+z^{-2}=9$, A, B, C being the points where the plane meets the rectangular axes of co-ordinates. Prove that the nine-points centre of ABC will lie on the surface

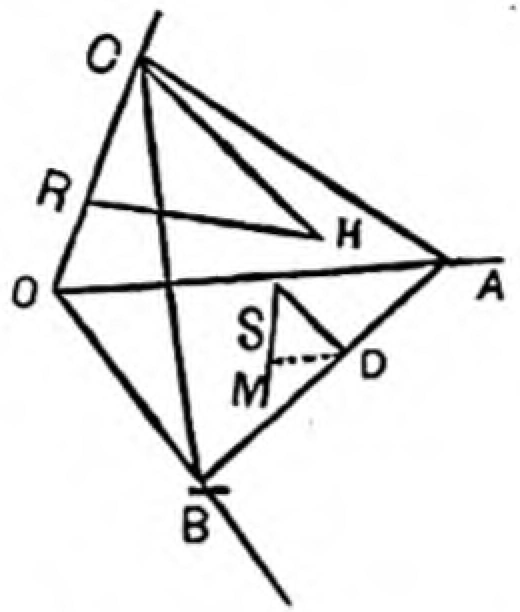
$$x\sqrt{4x^2-1}+y\sqrt{4y^2-1}+z\sqrt{4z^2-1}+2(x^2+y^2+z^2)=1.$$

Solution by K. J. Sanjana, M.A. and K. B. Madhava.

The co-ordinates of the centroid are evidently $\frac{1}{3}a$, $\frac{1}{3}b$, $\frac{1}{3}c$; since it lies on the surface,

$$\left(\frac{a}{3}\right)^{-2}+\left(\frac{b}{3}\right)^{-2}+\left(\frac{c}{3}\right)^{-2}=\left(\frac{1}{3}\right)^{-2},$$

$$\therefore \qquad \sum \frac{1}{a^2} = 1 \qquad \dots \qquad \dots \qquad \dots \qquad \dots \quad (1)$$



For the circumcentre S, we have

$$(x-a)^2+y^2+z^2=x^2+(y-b)^2+z^2=x^2+y^2+(z-c)^2,$$

along with the equation of the plane.

Thus we get z or $SM = \dfrac{c^2-1}{2c}$; $y = \dfrac{b^2-1}{2b}$, $x = \dfrac{a^2-1}{2a}$.

If H is the orthocentre, and HR parallel to the xy—plane meets OC in R, the $\triangle$S HCR and SDM are similar, where SD is perpendicular to AB.

Hence, since $CH = 2SD$,

$CR = 2SM$; and OR, or z for H, is

$$c - 2\frac{c^2-1}{2c} = \frac{1}{c}.$$

Thus the co-ordinates of the nine-points centre are given by

$$x = \tfrac{1}{2}\left\{\frac{1}{a} + \frac{a^2-1}{2a}\right\} = \frac{a^2+1}{4a}\ ;\ y = \frac{b^2+1}{4b},\ z = \frac{c^2+1}{4c}.$$

From these $a^2 - 4\,ax + 1 = 0$, &c. ;

$\therefore\ a = 2x \pm \sqrt{4x^2-1}$, and $\dfrac{1}{a} = 2x \pm \sqrt{4x^2-1}$, &c.

Substituting in condition (1) we get at once

$$\sum(8x^2-1) \mp \sum 4x\sqrt{4x^2-1} = 1,$$

or $$\sum 2x^2 \mp \sum x\sqrt{4x^2-1} = 1,$$

which is the equation of the locus.

QUESTIONS FOR SOLUTION.

808. (K. J. SANJANA, M. A.) :—Prove that

$$\frac{1}{y(y+z)}+\frac{1}{(y+1)(y+z+1)}+\frac{1}{(y+2)(y+z+2)}+\ldots ad\ \text{inf.}$$
$$=\frac{1}{y}-\frac{1}{2}\,\frac{z-1}{y(y+1)}+\frac{1}{3}\,\frac{(z-1)(z-2)}{y(y+1)(y+2)}-\frac{1}{4}\,\frac{(z-1)(z-2)(z-3)}{y(y+1)(y+2)(y+3)}+\ldots,$$

when y and z are positive rational numbers. Examine the identity for negative values of z.

809. (T. P. TRIVEDI, M. A., LL.B.) :—With the usual notation for an ellipsoid, and writing

$$\frac{1}{p}=\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2},$$

shew that

$$\int\frac{x+y+z}{p}\,dS=\frac{\pi}{16}abc\left\{(a+b+c)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right\}$$

the integral extending over that part of the ellipsoid for which x, y, z are all positive.

Write down also the value of

$$\int\frac{(x+y+z)^2}{p}\,dS \text{ in a symmetrical form.}$$

810. (T. P. TRIVEDI, M. A., LL.B.) :—Find the values (other than zero) which satisfy the equations :

$$x^2=y-z,\ y^2=z-x,\ z^2=x-y.$$

811. (C. KRISHNAMACHARY) :—Sum to n terms.

$$\text{(i)}\ 1+\frac{a^2}{(b+x)^2}\left(1+\frac{2x}{a+b}\right)+\frac{a^2(a+x)^2}{(b+x)^2(b+2x)^2}\left(1+\frac{4x}{a+b}\right)$$
$$+\frac{a^2(a+x)^2(a+2x)^2}{(b+x)^2(b+2x)^2(b+3x)^2}\left(1+\frac{6x}{a+b}\right)+\ldots$$

$$\text{(ii)}\ \frac{1}{y-x}+\frac{2(y+x)}{(y-x)(y-2x)}+\frac{3(y+x)(y+2x)}{(y-x)(y-2x)(y-3x)}+\ldots\ldots$$

812. (S. KRISHNASWAMI IYENGAR):—If S_n represents the sum of the reciprocals of the first n odd numbers prove that

$$\sum_1^\infty\frac{S_n}{4^n(n\,!)^2}=\sum_1^\infty\frac{1}{n^2}.$$

813. (S. KRISHNASWAMI IYENGAR) :—Show how to find the sum of the series

$$1+\cos\alpha+\cos 2\alpha\,\frac{2^n}{2\,!}+\cos 3\alpha\,\frac{3^n}{3\,!}+\ldots\ldots$$

and prove that when $n=2$, the sum of the series is

$$\{\cos(\sin\alpha)(\cos\alpha+\cos 2\alpha)-\sin(\sin\alpha)(\sin\alpha+\sin 2\alpha)\}\,e^{\cos\alpha}.$$

814. (K. J. SANJANA, M. A.) :—A regular quindecagon is inscribed in the circle $x^2+y^2=1$, with one angular point on the axis of x. Prove that the special vertices of the polygon (*i.e.*, those not common to it and the inscribed regular pentagon or triangle) lie at the intersection of the circle with the two hyperbolas

$$2(x^2-y^2)-x\pm\sqrt{15}y-2=0.$$

Prove also that the special vertices of the similarly inscribed polygon of 21 sides lie on the two cubics

$$(4x-1)(x^2-y^2)-4x+2\sqrt{7}\,xy=0.$$

815. (MARTYN M. THOMAS, M. A.) :—If

$$\int\frac{dx}{e^{ax^2+bx+c}}=\sum\frac{x^n}{n}k_n,$$

prove that $2ak_{n-1}+bk_n+nk_{n+1}=0$.

816. (MARTYN M. THOMAS, M. A.) :—If k be the curvature of an ellipse, and

$$a_0=\frac{1}{2\,!}k,\ a_1=\frac{1}{3\,!}\frac{dk}{ds},\text{a}\ \ a_2=\frac{1}{4\,!}\frac{d^2k}{ds^2},$$

show that $a_0^{-\frac{8}{3}}(4a_0^4-5a_1^2+4a_0a_2)$ has the constant value $\left(\dfrac{2}{\alpha\beta}\right)^{\frac{2}{3}}$ at all points of the ellipse, α and β being the semi-axes.

817. (S. MALHARI RAO.) :—If xC_0 is equal to the continued product of three primes whose sum is 327, find x.

818. (S. MALHARI RAO) .—Complete the following magic squares by inserting prime numbers in vacant cells :—

(*a*)

	1	
13		
		7

(*b*)

			47
		19	
	61		
53	83	1	43

(*c*)

		1		
		3		
103	97	449	43	37
		137		
		139		

819. (K. APPUKUTTAN ERADY) :—If ρ, ρ' be the radii of curvature at corresponding points of a curve and its α-evolute where α is a function of the arc measured from a fixed point on the curve, show

$$\rho'\left(\frac{1}{\rho}+\frac{d\alpha}{ds}\right)+\sin\alpha=\frac{d}{ds}\left\{\frac{\cos\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}.$$

820. (K. APPUKUTTAN ERADY) :—Prove that the equation

$$(r+r_1)(r-r_1)^2=4a(ar+r_1-\lambda rr_1)$$

in bipolar co-ordinates, represents the lines of force of a simple magnet, and show that the system of orthogonal curves is given by the equation

$$\frac{1}{r}-\frac{1}{r_1}=\mu.$$

821. (M. K. KEWALRAMANI) :—If through A, B, C lines AXY, BYZ, CZX are drawn so as to make the same angle θ with AB, BC, CA respectively, and form the triangle XYZ, prove that

$$\rho=2\sigma\sin(\omega-\theta),$$

where ω is the Brocard angle of the triangle ABC, σ is radius of the First Lemoine Circle of ABC, and ρ is the radius of the Cosine Circle of the triangle XYZ.

822. (K. K. RANGANATHA AIYAR, M. A.) :—If a circle cut an ellipse in A, B, C, D whose eccentric angles are α, β, γ, δ then the power of any point P of the ellipse varies as

$$\sin\tfrac{1}{2}(\theta-\alpha).\sin\tfrac{1}{2}(\theta-\beta)\times\sin\tfrac{1}{2}(\theta-\gamma)\sin\tfrac{1}{2}(\theta-\delta).$$

Hence or otherwise show that the equation to an ellipse may be put in the form $S_1^{\frac{1}{3}}+S_2^{\frac{1}{3}}+S_3^{\frac{1}{3}}=0$, where ABC is a maximum inscribed Δ and S_1, S_2, S_3 the circles of curvature at A, B, C.

823. (K. K. RANGANATHA AIYAR, M. A.) :—If P_1 denote the parabola of closest contact with an ellipse at A, we may put the equation to the ellipse in the form $P_1^{\frac{1}{4}}+P_2^{\frac{1}{4}}+P_3^{\frac{1}{4}}=0$, ABC being any inscribed Δ.

824. (F. H. V. GULASEKARAM, B.A.) :—With reference to the circle of curvature of the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ at the vertex A of the triangle of reference, prove the following :—

(i) ρ (the radius of curvature)

$$=\frac{Rabc}{lmn}\left[\frac{m^2+n^2-2mn\cos A}{a^2}\right]^{3/2}$$

(ii) The perpendicular distances β, γ of the centre of curvature from the sides CA, AB are given by the equations

$$\frac{\beta}{n-m\cos A}=\frac{\gamma}{m-n\cos A}=\frac{\rho}{(m^2+n^2-2mn\cos A)^{\frac{1}{2}}}$$

(iii) The equation to the radical axis of the circle of curvature and the circumcircle of the triangle of reference is

$$\frac{\beta/b+\gamma/c}{\beta/m+\gamma/n}=\frac{m^2+n^2-2mn\cos A}{al}$$

825. (ENQUIRER).—Salmon [*Conic Sections*, Appendix on Pascal's theorem] states that the two of points denoted by

$$\left\{\begin{matrix} ab,\ de,\ cf \\ cd,\ fa,\ be \\ ef,\ bc,\ ad \end{matrix}\right\} \text{ and } \left\{\begin{matrix} ab,\ cd,\ ef \\ de,\ fa,\ bc \\ cf,\ be,\ ad \end{matrix}\right\}$$

are conjugate with respect to the conic. Prove that this is so.

826. (A. C. L. WILKINSON):—If P_1, P_2, P_3, P_4, P_5, P_6, are six points on a conic and $\lambda=(123456)$, $\mu=(126453)$, $\nu=(123546)$, $\rho=(126543)$ are four intersecting Pascal lines corresponding to hexagons taken in the orders of the suffices as stated, prove that the cross-ratio $(\lambda\mu,\ \nu\rho)$ is equal to

$$\frac{(32,\ 16)(31,\ 26)}{(34,\ 56)(35,\ 46)}$$

where $(pq,\ rs)$ denotes the cross-ratio of the pencil subtended by the points P_p, P_q, P_r, P_s at any point of the conic.

827. (A. C. L. WILKINSON):—If a skew surface is defined by

$$x=x,\ y=bx+\beta,\ z=cx+\gamma,$$

where b, c, β, γ are functions of t, and if the axis of x is the generator corresponding to $t=0$, the origin the central point of this generator, and $z=0$ the tangent plane at the origin; then the hyperboloid of closest contact along the generator $t=0$ is given by

$$2c'\beta'^2z=c'\gamma''y^2+2c'^2\beta'xy-b''\beta'z^2-(c'\beta''-c''\beta')yz$$

where the values of the differential coefficients of b, c, β, γ are for $t=0$.

List of Periodicals Received.

(From 16th July to 15th September 1916.)

1. Annals of Mathematics, Vol. 17, No. 4, June 1916.
2. Astrophysical Journal, Vol. 43, Nos. 4 & 5, May and June 1916.
3. Bulletin of the American Mathematical Society, Vol. 22, Nos. 9 & 10 June, July & August 1916.
4. Bulletin des Sciences Mathematics, Vol. 40, May, June & July 1916.
5. L'Intermediaire des Mathematiciens, Vol. 23, Nos. 5 & 6, May & June 1916.
6. Liouville's Journal, Vol. 1, No. 4.
7. Mathematical Gazette, Vol. 8. Nos. 123 and 124, May and July 1916 (3 Copies).
8. Mathematical Questions and Solutions, Vol. 2, Nos. 1 & 2, July & August 1916. (5 Copies).
9. Mathematics Teacher, Vol. 8, No. 4, June 1916.
10. Messenger of Mathematics, Vol. 45 and 46 Nos. 12 and 1, April and May 1916.
11. Philosophical Magazine, Vol. 32, Nos. 187 and 188, July and August 1916.
12. Popular Astronomy, Vol. 24, Nos. 6 & 7, June, July, August and Septem- 1916. (3 Copies).
13. Proceedings of the London Mathematical Society, Vol. 15, No. 4, August 1916.
14. Proceedings of the Royal Society of London, Vol. 92, No. 642 July 1916.
15. Quarterly Journal of Mathematics, Vol. 47, No. 2, July 1916.
16. Transactions of the American Mathematical Society, Vol. 17, No. 3 July 1916.
17. Transactions of the Royal Society of London, Vol. 216, Nos. 545, 546, 547 and 548 and Vol. 217, Nos. 549 and 550.
18. The Tohoku Mathematical Journal, Vol. 9, No. 4, June 1916, Vol. 10, Nos. 1 and 2, August 1916.
19. American Mathematical Monthly, Vol. 23, Nos. 5 and 6, May and June 1916.

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